Vancouver Geotechnical Society

A Local Section of the Canadian Geotechnical Society

www.v-g-s.ca

Vancouver Geotechnical Society Workshop

Critical State Soil Mechanics: Notes

By Mike Jefferies, PEng and Dr. Dawn Shuttle, PEng

(geomek@hotmail.com dawn_shuttle@hotmail.com)

Rev. 9: Feb 2020

Overview

Geotechnical engineering characterizes soil behaviour using ϕ, s_u, C_c and so forth, with these parameters measured using insitu and/or laboratory tests. Advanced (or 'theoretical') soil mechanics seeks to understand these familiar parameters in terms of fundamental properties. A feature of advanced soil mechanics is use of physical principles rather than correlations; an immediate benefit is being able to compute, for example, how changes in conditions (for example void ratio and/or stress level) affect the geotechnical parameters such as ϕ.

Recent dam failures, with transitions from drained to undrained conditions in tens of seconds (or quicker), have challenged conventional geotechnical engineering which provides little in the way of explaining how such failures develop. But, this mechanism is very simple in terms of physical principles. It is these sudden dam failures, possibly more than anything else, that have created interest in what advanced soil mechanics offers practical engineers – and thus this course.

Advanced soil mechanics applies the doctrine 'if you cannot compute, you have nothing'. This phrase is more important than it looks, as it derives from a most fundamental question about knowledge (search the *Church–Turing thesis* on Wikipedia). For the practical engineer it amounts to requiring that our theories 'add up'; and the easiest way to find that out is to use a spreadsheet – which is the approach followed in this course. Properties are derived from data, but then these properties are used to formally compute the soil's behaviour in laboratory tests – so checking that what you believe to be your 'understanding' is actually consistent with the raw measurements you made.

Traditional views and teaching of advanced soil mechanics divide the subject into pore water movement ('consolidation' theories) and strength (constitutive modelling). This course only considers strength and stiffness, with pore water being treated as either drained or undrained. The work hardening theory of plasticity is used, in the form developed as *critical state soil mechanics* (CSSM) after the title of the book that gave the first complete exposition of the approach. The models used are *Original Cam Clay* and *NorSand*; these names do not denote a restriction to a soil type but are best thought of as simply brand names for a particular collection of equations.

The course comprises four modules. The first module is about soil behaviour seen in laboratory tests – there is little theory as such here other than the idea that soil comprises particles. The second module looks to the ideas about how soil behaviour can be represented in a computable manner albeit a little idealized. The third module is about a proper generalization of the theory that works for all soils under all conditions. So far everything has been focused on laboratory tests where we know the soil's void ratio and type – which you likely will not know insitu. The fourth module is about the piezometric cone penetration test (CPTu) which as well as giving all sorts of stratigraphic information is the most accurate measurement of insitu soil state and which allows laboratory-based understanding to be used in practical situations.

Lectures are interwoven with tutorials, as that is the most effective way of learning. You will need to be familiar with Excel and making graphs within Excel.

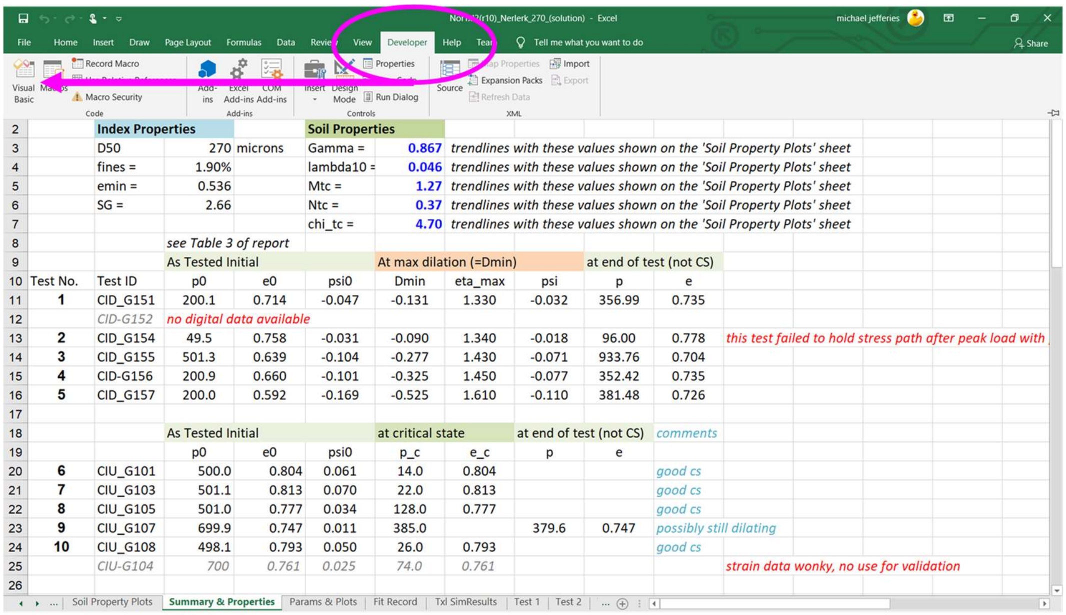
Analytical methods (FLAC, Plaxis, ABAQUS etc) are not dealt with in this course. Rather, this course aims to give you the tools to obtain realistic inputs for such analyses.

Software Requirements

The course is going to use Excel for nearly everything. Initially, we will be using worksheets (with templates provided to save time in setting things up); this is perhaps the easiest way to appreciate how things like stress-dilatancy come together with the hardening law. However, when you start using the theory as an engineer you will find that you need to quickly look at various tests to determine soil properties. And, as we move from ‘learning’ to ‘real soils’, we need to remove a few simplifying assumptions – and which causes very cluttered, and difficult to read formulae, if you stay within a worksheet. Microsoft has anticipated this situation and provides a proper programming language, Visual Basic for Applications (VBA), that lies behind Excel and is accessed from the standard Excel menu bar.

The course will use VBA for soil property determination and CPT processing. This software will be provided, as by the time we switch to VBA you will have already understood how CSSM works. There are books on programming in VBA, but what you need is really only to make yourself familiar with the provided code – and it is all written in ‘plain English’. If you would like to know more about VBA, the book *Excel VBA Programming for the Absolute Beginner* by Birnbaum & Vine is good and easily found on Amazon. You do not need it for the course, but it may be useful to you in the future as VBA is a truly wonderful feature within Excel for engineering.

Please check that Excel on the laptop you bring to the CSSM workshop has got VBA enabled. You can check this as shown on the piccy below under the Excel menu (the shortcut is to press ‘Alt’ = ‘F11’ keys together).



2		Index Properties			Soil Properties						
3		D50	270	microns	Gamma =	0.867	trendlines with these values shown on the 'Soil Property Plots' sheet				
4		fines =	1.90%		lambda10 =	0.046	trendlines with these values shown on the 'Soil Property Plots' sheet				
5		emin =	0.536		Mtc =	1.27	trendlines with these values shown on the 'Soil Property Plots' sheet				
6		SG =	2.66		Ntc =	0.37	trendlines with these values shown on the 'Soil Property Plots' sheet				
7					chi_tc =	4.70	trendlines with these values shown on the 'Soil Property Plots' sheet				
8			see Table 3 of report								
9			As Tested Initial			At max dilation (=Dmin)			at end of test (not CS)		
10	Test No.	Test ID	p0	e0	psi0	Dmin	eta_max	psi	p	e	
11	1	CID_G151	200.1	0.714	-0.047	-0.131	1.330	-0.032	356.99	0.735	
12		CID-G152	no digital data available								
13	2	CID_G154	49.5	0.758	-0.031	-0.090	1.340	-0.018	96.00	0.778	this test failed to hold stress path after peak load with
14	3	CID_G155	501.3	0.639	-0.104	-0.277	1.430	-0.071	933.76	0.704	
15	4	CID-G156	200.9	0.660	-0.101	-0.325	1.450	-0.077	352.42	0.735	
16	5	CID_G157	200.0	0.592	-0.169	-0.525	1.610	-0.110	381.48	0.726	
17											
18			As Tested Initial			at critical state		at end of test (not CS)		comments	
19			p0	e0	psi0	p_c	e_c	p	e		
20	6	CIU_G101	500.0	0.804	0.061	14.0	0.804			good cs	
21	7	CIU_G103	501.1	0.813	0.070	22.0	0.813			good cs	
22	8	CIU_G105	501.0	0.777	0.034	128.0	0.777			good cs	
23	9	CIU_G107	699.9	0.747	0.011	385.0		379.6	0.747	possibly still dilating	
24	10	CIU_G108	498.1	0.793	0.050	26.0	0.793			good cs	
25		CIU-G104	700	0.761	0.025	74.0	0.761				strain data wonky, no use for validation
26											

VBA should be there automatically with most installations of Office, but just in case please check and if not there get your systems folks to set it up for you before the course. Excel treats VBA as a ‘macro’, so sometimes it will trigger an *Enable Macros?* warning when opening files because various idiots have written viruses using macros; if your system folks want to check what is there, everything in the provided programs is ‘open code’ without password protection (ie all is immediately visible).

Schedule

Note: timings are approximate and will depend on progress of participants during the workshop. If need be, will extend the workshop until all participants have accomplished all steps (the largest stumbling block is normally the initial OCC spreadsheet, so any needed update to the schedule should be apparent by Friday morning).

Thursday 6 February 2020

Time	Topic	Comments
0800 – 0830	Registration	
0830 – 0845	Introduction	Overview of the course and learning goals
0845 – 0945	1. Soil strength behaviour	Lecture: Bishop-Taylor strength model; as proposed and modern implementation. Casagrande critical state and Taylor's extension; modern CSL methods. State parameter control of dilatancy.
0945 – 1000	Coffee Break	Provided
1000 – 1115	**TUTORIAL**. Determine soil properties using provided triaxial data.	Numerical differentiation (in Excel). Using provided data: - develop the CSL using Γ, λ - determine strength properties M, N - determine dilatancy property X
1115 – 1130	2a. Theory of Plasticity	Lecture: Nature of plasticity: Yield surfaces, normality, hardening. Necessary form of plasticity for soils.
1130 – 1230	2b. Original Cam Clay (OCC)	Lecture: Idealizations used in OCC. Derivation of OCC equations.
1230 – 1330	Lunch Break	
1330 – 1430	2c. Numerical integration procedure	Lecture: Euler integration and introduction to the xls template used for the OCC tutorials.
1430 - 1600	**TUTORIAL**. Implement OCC for undrained triaxial tests and verify	Using the provided template, code OCC into Excel for a CIU test. Verify the xls by comparison to Schofield & Wroth closed-form solution.
1515 -1530	Coffee break during tutorial	
1530 - 1600	**Continue with tutorial…**	***This is the most difficult part of the course !***
1600 – 1615	2d. From undrained to drained, with validation against test data	Lecture: How to implement drained loading paths with OCC; test data provided for validation.
1615 - 1700	**TUTORIAL**. Drained triaxial tests using OCC.	Change the verified OCC xls to drained loading and compare with test data on a loose sand to confirm that OCC 'works' for some real soils.

Friday 7 February 2020

Time	Topic	Comments
0830 – 0900	2e. Limitations of OCC	Lecture. Why OCC does not match real soil behaviour most of the time.
0900 – 1000	3. Fixing OCC for real soils => NorSand (NS)	Lecture. Axioms; state parameter; over-consolidation; image condition; hardening limit.
1000 – 1015	Coffee Break	Provided
1015 – 1115	**TUTORIAL**. Implement NS in xls worksheet	Change working OCC xls to NS; validate NS against dense sand data (provided in template)
1115 – 1200	4. Elasticity	Lecture. Shear and bulk moduli; bender and VSP methods; Poisson's ratio for soils; elastic models; the geophysical conundrum
1200 – 1230	5a. Moving from worksheets to VBA; NS implementation in VBA.	Lecture: proper engineering process using Excel and VBA; intro to VBA; introduction to the VBA; aspects of NS VBA implementation in the provided *NerlerkTxl.xls* template.
1230 – 1330	Lunch	
1330 – 1430	**TUTORIAL**. Calibration of NS to Nerlerk sand test data	Use Nerlerk properties determined in first tutorial to fit NS to the 5xCID + 5xCIU data provided. Determine the H property.
1430 – 1445	5b. The transition form drained loading to undrained static liquefaction.	Guided tutorial using calibrated NerlerkTxl.xls.
1445-1515	6. Measuring ψ in situ using the CPT. Procedures, processing, and presentation of results	Lecture. Important aspects of CPT investigations, methods for evaluating CPT data, and introduction to *CPTplot.xls*
1515 - 1530	Coffee break	Provided
1530 – 1600	**TUTORIAL.** Using *CPTplot.xls*	Become familiar with the options and procedures in CPTplot.xls. ***Participants are welcome to use data from one of their own CPT soundings*** (as long as the data can be imported into xls)
1600 – 1700	7a. Getting ψ from CPT data in *sands* (= drained penetration). Introduction to *CPTwidget.exe*	Lecture. Calibration chambers; universal trends; effect of soil properties; importance of Gmax. CPT coefficient determination using finite element program *CPTwidget.exe.* Input of computed calibration into *CPTplot.xls*
1700 - 1800	Attitude adjustment hour	

Saturday 8 February 2020

Time	**Topic**	**Comments**
0830 – 0930	**TUTORIAL.** Using *CPTwidget.exe* for drained CPT soundings	Participants run the "widget" to calibrate CPT for a set of sand properties to determine the inversion coefficients *k,m;* input computed calibration to *CPTplot*
0930 – 1000	7b. Getting ψ from CPT data in *silts* (= undrained penetration).	Lecture. Capturing effect of measured induced excess pore pressure at u_2; changes to *CPTwidget* inputs; calculation of N_{kt} factor for chosen soil properties. Input of computed calibration into *CPTplot.xls;* Cadia validation
1000 – 1015	Coffee Break	Provided
1015 - 1100	**TUTORIAL.** Processing results from *CPTwidget* for undrained penetration.	Use provided widget output to generate a CPT calibration in silt. Investigate s_r, s_u, brittleness index on liquefaction. Input calibration into *CPTplot* and explore results
1100 – 1130	7c. Plewes Method	Lecture. Basis of method; Reid update; site-specific adjustment; limitations and advantages
1130 - 1230	8. Sands to Silts - differences in behaviour, similarities in behaviour	Lecture. CSL measurement and repeatability with tailings; laboratory testing issue; frictional properties; Gmax
1230 – 1330	Lunch	
1330 – 1415	9. From triaxial to plane-strain and beyond	Lecture. The theory looks after you.
1415 – 1445	10. Summary of workshop	Review of key 'learnings'; resources for participants in going forward.
1445 – 1600	Open discussion and additional tutoring as requested	
1600 sharp !	Participants buy beer for course tutors…	

Course Data Files and Templates

The course is based on participants working on their own laptops in a tutorial setting (some people find it better to work in pairs). There are several files that go with the course and which participants need to copy/download to their computers, as these files will be used during the tutorials. Files are all included under a single directory '*CSSM_course_data*' and which contains the following folders:

Folder (sub-directory)	**Contents**
Light Reading	Papers about aspects of soil mechanics history; also a copy of the S&W book. This is background information for those interested, and is not required reading.
Course Notes	These course notes and schedule for the workshop
Templates for Exercises	Excel templates for: • Tutorial 1 (determining sand properties) • Tutorial 2 (implementing Original Cam Clay) Tutorials 3 and 4 will use the completed xls developed in Tutorial 2. Tutorial 5 will use the template of Tutorial 1 together with the sand calibration developed during Tutorial 1
CPT Programs	Two sub-directories containing: • *CPTplot.xls*, the processing and plotting program; includes a user manual. • *CPTwidget*, including program notes, source code, executable, example inputs, and xls for developing the calibration

These course notes cover the situation in the laboratory where void ratio is known. Practically, engineers must determine void ratio in the ground – a topic that has an extensive literature, much of which is covered in Chapter 4 of the *Soil Liquefaction* book. A convenient and easy to read article on using the CPT is the paper *Determining Silt State from CPTu* which can be downloaded (it is free) from the *Geotechnical Research Journal* at https://www.icevirtuallibrary.com/doi/full/10.1680/jgere.16.00008. This particular paper describes how the *CPTwidget* was developed; the version of *CPTwidget* in the course downloads is a further development with outputs of pore pressure at the CPT's u_2 sensor location and the computed N_{kt} factor – both useful and important enhancements.

Notation

Subscripts

c	Critical state
i	Image condition (occurs when $D^p \equiv 0 \wedge \dot{D}^p \neq 0$)
tc	Triaxial compression condition ($\theta = \pi/6$)
h	Horizontal
v	Vertical; volume
0	Initial condition
1, 2 ,3	Principal directions of stress or strain

Superscripts

e	Elastic
p	Plastic
Dot "·"	Denotes increment

Stress Variables (bar over or ′ denotes effective)

$\sigma_{1,2,3}$	[FL^{-2}]	Principal stresses
$\bar{\sigma}_m$	[FL^{-2}]	Mean effective stress $\bar{\sigma}_m = (\bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_3)/3$
$\bar{\sigma}_q$	[FL^{-2}]	Deviatoric stress invariant $\bar{\sigma}_q = (\tfrac{1}{2}(\sigma_1 - \sigma_2)^2 + \tfrac{1}{2}(\sigma_2 - \sigma_3)^2 + \tfrac{1}{2}(\sigma_3 - \sigma_1)^2)^{\frac{1}{2}}$
p'	[FL^{-2}]	Mean effective stress (= $\bar{\sigma}_m$)
q	[FL^{-2}]	Triaxial deviator stress. $q = \sigma_1 - \sigma_3$ (= $\bar{\sigma}_q$)
η	[-]	Dimensionless shear measure as ratio of stress invariants $\eta = \bar{\sigma}_q / \bar{\sigma}_m$
θ	[Rad]	Lode angle, $\sin(3\theta) = -13.5\bar{\sigma}_1\ \bar{\sigma}_2\ \bar{\sigma}_3 / \bar{\sigma}_q^3$
α	[Rad]	Included angle between direction of major principal stress (the "1" direction) and coordinate frame of reference
u	[FL^{-2}]	Pore pressure

Strain Variables (dot superscript denotes increment)

$\varepsilon_{1,2,3}$	[-]	Principal strains (assumed coaxial with principal stresses)
$\dot{\varepsilon}_v$	[-]	Volumetric strain $\dot{\varepsilon}_v = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3$
$\dot{\varepsilon}_q$	[-]	Shear strain measure work conjugate with $\bar{\sigma}_q$ $\dot{\varepsilon}_q = \frac{1}{3}\left((\sin\theta + \sqrt{3}\cos\theta)\dot{\varepsilon}_1 \ - 2\sin\theta\ \dot{\varepsilon}_2 + \ (\sin\theta - \sqrt{3}\cos\theta)\dot{\varepsilon}_3\right)$
D^p	[-]	Plastic dilatancy, as strain rate ratio $\dot{\varepsilon}_v^p / \dot{\varepsilon}_q^p$

State Variables

e	[-]	Void ratio
K_0	[-]	Geostatic stress ratio, $K_0 = \bar{\sigma}_h / \bar{\sigma}_v$
ψ	[-]	State parameter, $\psi = e - e_c$
R	[-]	Over-consolidation ratio $R = p'_{\max} / p'$
χ	[-]	Scaling factor (soil property) for state dilatancy

Elasticity

E	[FL^{-2}]	Young's modulus
G	[FL^{-2}]	Shear modulus
K	[FL^{-2}]	Bulk (volumetric) modulus
I_r	[-]	Soil shear rigidity $(= G/\bar{\sigma}_m)$
κ	[-]	Slope of elastic line in e-ln(σ_m) space
ν	[-]	Poisson's ratio

Critical State

Γ	[-]	Reference void ratio on CSL, defined at p′ = 1 kPa
λ	[-]	Slope of CSL in e-ln(σ_m) space for semi-log idealization
λ_{10}	[-]	Slope of CSL, but defined on base 10 logarithms (= 2.3 λ)
M	[-]	Critical friction ratio, equals η_c at the critical state. Varies with Lode angle, value at triaxial compression (M_{tc}) taken as reference soil property.

CPT Parameters and Variables

q_t	[FL^{-2}]	CPT tip resistance after correction for unequal end area
f_s	[FL^{-2}]	CPT friction sleeve stress measurement
u_c	[FL^{-2}]	Pore pressure measured by CPT during sounding at shoulder location (sometimes denoted as u_2 location in the literature).
Q	[-]	Dimensionless CPT resistance based on vertical stress. Corresponds to standard usage within *in situ* testing community, $Q = (q_t - \sigma_{v,0}) / \bar{\sigma}_{v,0}$
Q_p	[-]	Dimensionless CPT resistance based on mean stress, $Q_p = (q_t - p_0) / p'_0$
B_q	[-]	CPT$_u$ excess pore pressure ratio based on excess pore pressure measured at 'u2' location: $B_q = (u_2 - u_0)/(q_t - \sigma_{v0})$
F	[-]	Stress normalised CPT friction ratio: $F = f_s/(q_t - \sigma_{v0})$
k, m	[-]	Soil property and rigidity specific coefficients in equation relating Q_p to ψ

Soil Strength and Dilation

Soil behaviour, across the spectrum from gravels to clays, is controlled by effective stress. The details of soil behavior are most directly seen in drained tests and which leads to a preference for understanding the physics using sands simply because drained tests in clay take a lot of time. Thus, this section largely looks to sand data supplemented by some data on clays. Undrained response will be dealt with in the following section when we involve theory.

Soil is Particulate

Scoop up some sand on a beach and look at it in your hand – you can see particles, not a solid material. If you use a microscope, the nature of the particles is even clearer. If we move to finer soil the eye is no longer enough and we may resort to a scanning electron microscope for silts, see Figure 1, and observe a collection of individual particles of different sizes.

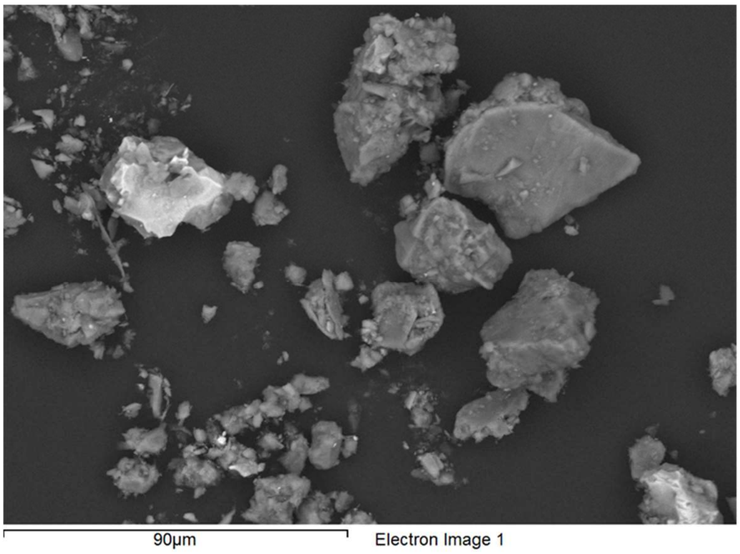


Figure 1: Particle shapes in a SILT seen with a scanning electron microscope

A collection of equal sized spheres sketched on Figure 2 was suggested as a first idealization of soil more than a century ago (Reynolds, 1885). This idealization leads to fundamental concepts of how and why soil behaves as it does. If a collection of equal sized spheres is in its densest packing (body centered cubic, right hand side of Figure 2) then any distortion of the assembly will cause the assembly to expand: dilatancy. Conversely, if the collection is in its loosest packing (face centered cubic, left hand side of Figure 2) then any distortion will allow the particles to pack closer together: also a form of dilation, albeit in the opposite sense. These are 'kinematic' ideas associated with how the (supposedly rigid) particles must move to accommodate distortion of the assembly, with no theory as such. If we accept soil is particulate, and approximates spheres, as we hope all can agree simply from looking at soil (using a microscope if needed), then we must accept that dilation is going to be inextricably linked to soil behaviour.

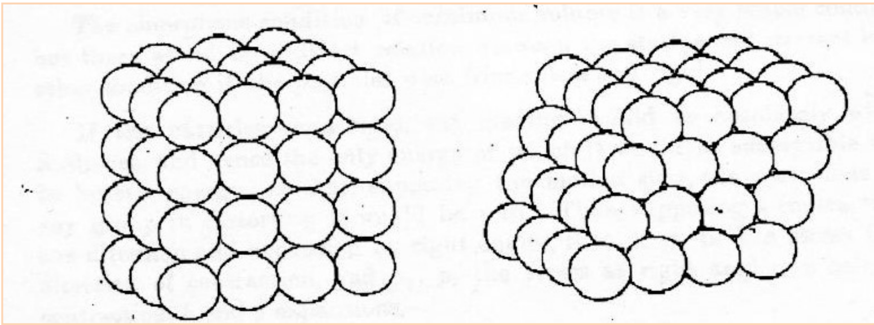


Figure 2: Reynolds (1885) kinematic explanation of dilation
The packing in either the left or right arrangement is controlled by the bounding spheres; and it either case distortion causes a change in volume of the assembly

Interlocking and True Friction

Early strength testing used direct shear as the equipment was simple, Figure 3. Example test results shown on Figure 3 illustrate the familiar Mohr-Coulomb strength criterion. The interesting question that then follows is the physical nature of the 'cohesion' intercept, since our photographs of soil particles (Figure 1) show there cannot be any bonds between the particles; so, the ideas in the 1940's developed primarily at Harvard (Casagrande) and MIT (Taylor), were that the 'cohesion' was related to particle 'interlocking' – and interlocking is the terminology you find in early publications about the nature of soil strength (eg Taylor, 1948).

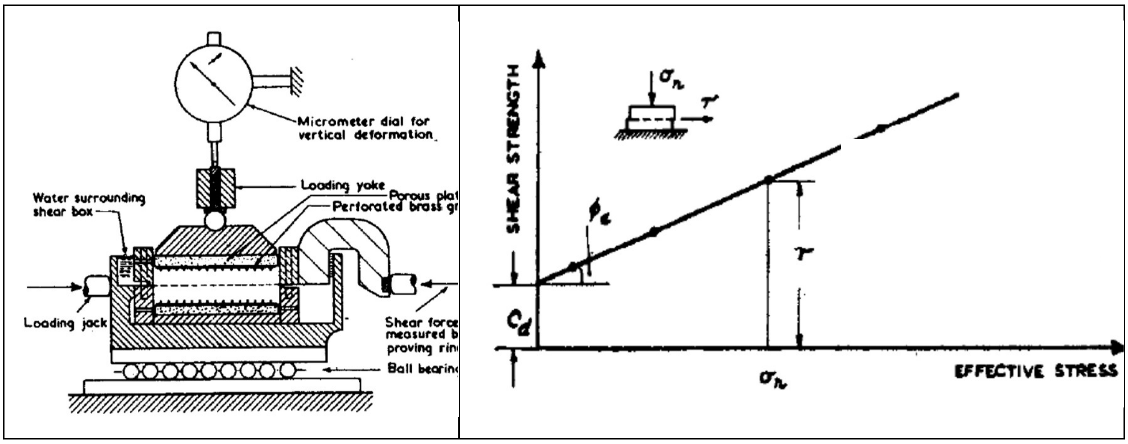


Figure 3: Shear box testing and example results (Skempton & Bishop, 1950)

The idea of strength alone is missing half the story. Looking at the results of a direct shear test make it quickly apparent the Reynold's idea about volume change during shear are correct – dense sand samples expand with the top cap rising during shear (note the micrometer dial in Figure 3). This led to the idea of 'energy corrections' since the top cap rising against the confining stress was clearly doing work – and thus that not all of the applied shear stress was being resisted by 'friction'. The corollary was that workers in the 1940-55 period developed interest in the nature of mobilized friction.

Taylor (1948, Sect. 14.9) provides an example of the importance of interlocking noting that it contributed 26% of the strength measured in a direct shear test on dense Ottawa sand (a standard laboratory soil). However, it was Bishop (1950) who first put the mathematics to the behaviour deriving, using considerations of incremental work, the strength equation (see Figure 4):

$$\frac{\tau}{\sigma_n} = \frac{\tau_f}{\sigma_n} + \frac{\delta v}{\delta \Delta} \quad [1]$$

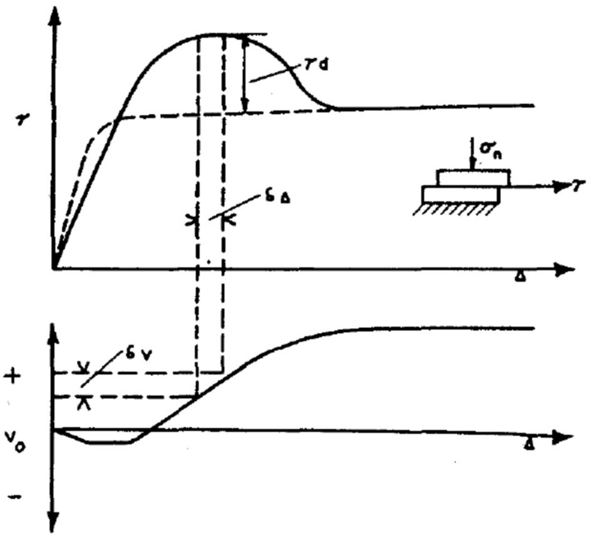


Figure 4: Soil strength related to rate of volume change in direct shear (Bishop, 1950)

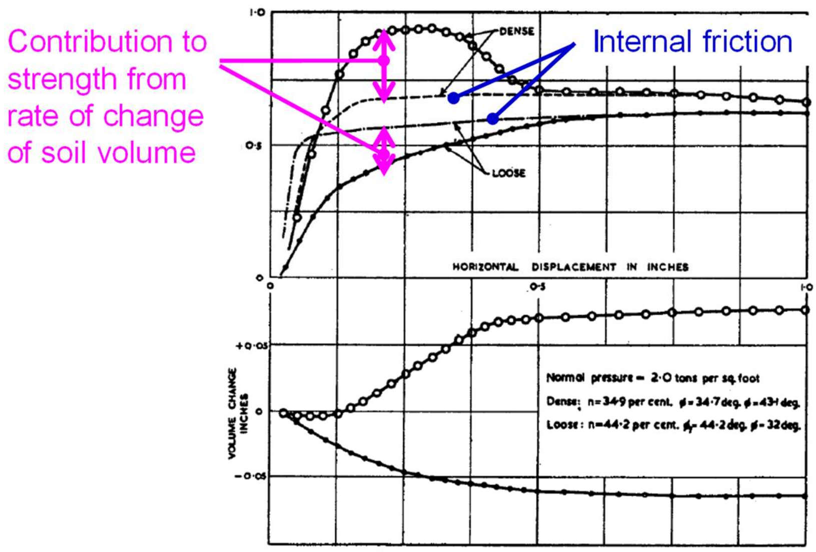


Figure 5: Evolution of soil strength components in direct shear (after Bishop. 1950)

As interest was in separating 'interlocking' from 'true friction', Bishop used equation [1] to compute τ_f for a loose and a dense direct shear test with the results shown on Figure 5. Notice how τ_f rapidly develops to a near-constant value, which is rather similar whether dense or loose. Also notice how the strength component from dilation evolves with shear strain – the contribution from rate of volume change with shear is not something applying at just peak strength. Sample expansion increases strength, as expected, but sample volume decrease decreases the strength; mathematically, the same equation controls and there is a complete spectrum of behaviour depending on how void ratio evolves. In essence, this is the behaviour anticipated from idealizing soil as a collection of particles as per Figure 2.

The maximum value of the stress ratio τ_f/σ_n was called the *tangent of the angle of internal friction*; there was a tacit implication that this internal friction was a true soil property unlike strength (strength depended on how much dilation developed).

These ideas of internal friction and dilation led to the 'sawtooth' idealization for soil strength, Figure 6, where the *internal friction* is that on the sliding tooth face while *dilation* is the tooth angle – a simple way of appreciating the basic behaviour.

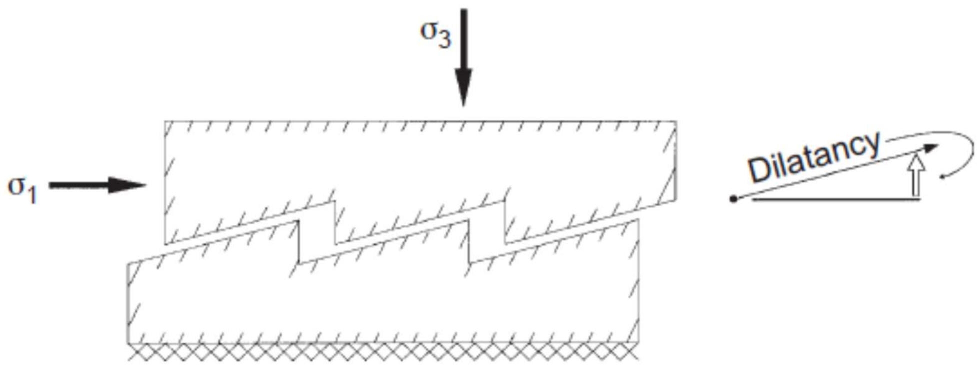


Figure 6: Sawtooth model for soil strength

From Direct Shear to Triaxial: Invariants

The direct shear test is mechanically simple but has two deficiencies: it is difficult to control drainage and associated measurement of pore water pressure; and, only half the stresses acting on the soil are measured. So, on one hand the direct shear test is not easy to use for exploring the spectrum of soil behaviour and on the other hand, even if that limitation is overcome, there are insufficient measurements to know the true stress state. The triaxial test suffers from neither of these problems and has become the reference test for soils. The classic reference on the triaxial test is Bishop & Henkel (1957) and that book still has much of value; however, today it is normal to use computer control and data acquisition rather than mechanical methods for loading and manually recorded gauges for soil response – have a look at the website www.gdsinstruments.com to see what is now possible and used in commercial 'good practice' testing.

As the triaxial test provides complete data on stresses and strains, it is now possible to improve on our understanding of soil behaviour. Metals deform at constant volume and confining stress has no effect on their behaviour; soils (= particulate materials) are more complicated in that their behaviour depends on confining stress (= 'frictional') and they change volume with changes in confining stress and changes in distortion (= dilatancy). The mechanics of soil behaviour are simplified if we recognize these processes and choose appropriate stress and strain measures.

The choice of stress and strain measures is also influenced by the principle that, for an isotropic material with no intrinsic sense of direction, it must not matter if we are looking from the left or from above when we define our stress and strain measures; this is the notion that we should use stress and strain *invariants* that have no sense of direction.

The triaxial test uses cylindrical samples and thus two stresses and two strains are always equal because of cylindrical symmetry. For the usual situation of standard triaxial compression loading $\sigma_2 = \sigma_3$ and $\varepsilon_2 = \varepsilon_3$. The stress and strain invariants for this condition of symmetry are shown on Table 1; note that all stresses are 'effective'.

Table 1: Stress and strain invariants for triaxial compression

	Distortion	Volume Change
Stress	$q = \sigma_1 - \sigma_3$	$p' = (\sigma_1 + 2\ \sigma_3) / 3$
Strain	$\varepsilon_q = {}^2/_3\ (\varepsilon_1 - \varepsilon_3)$	$\varepsilon_v = \varepsilon_1 + 2\ \varepsilon_3$

The '${}^2/_3$' factor appearing in the deviatoric strain invariant ε_q is there to make the stress and strain invariants work conjugate, which is just a fancy phase that the incremental work δW on the soil being given by:

$$\delta W = q\ \delta\varepsilon_q + p'\delta\varepsilon_v \qquad [2]$$

As a further wrinkle, note that the 'compression positive' convention of soil mechanics makes void ratio reduction correspond to positive volumetric strain: $\delta\varepsilon_v = -\ \delta e/\ (1+e)$.

The invariants shown on Table 1 generalize to 3D, but triaxial compression is sufficient for a practical engineer to understand soil behaviour and so we restrict this course to triaxial compression. These invariants are sometimes called 'Cambridge stress variables' (or equivalent phrase) in the literature, but that is a wrong attribution – these invariants were proposed as part of the mathematical theory of plasticity which much predates their adoption for soil mechanics.

Triaxial Compression Strength

The Taylor/Bishop strength relation developed considering direct shear, Equation [1], generalizes to triaxial compression. We do this by changing our parameters from those of a direct shear test to the invariants defined on Table 1 so that:

$\eta = M - D$ [3]

…where, by comparison with, [1]:

$\eta = q / p'$ is used instead of τ/σ_n [3a]

$D = \delta\varepsilon_v / \delta\varepsilon_q$ is used instead of $\delta v/\delta\Delta$ [3b]

This transformation in [3] is not quite enough. Figure 7 shows the results from many drained triaxial tests on Brasted sand, with the tests ranging from loose to dense, at peak strength (ie plotting η_{max} vs D at η_{max}).

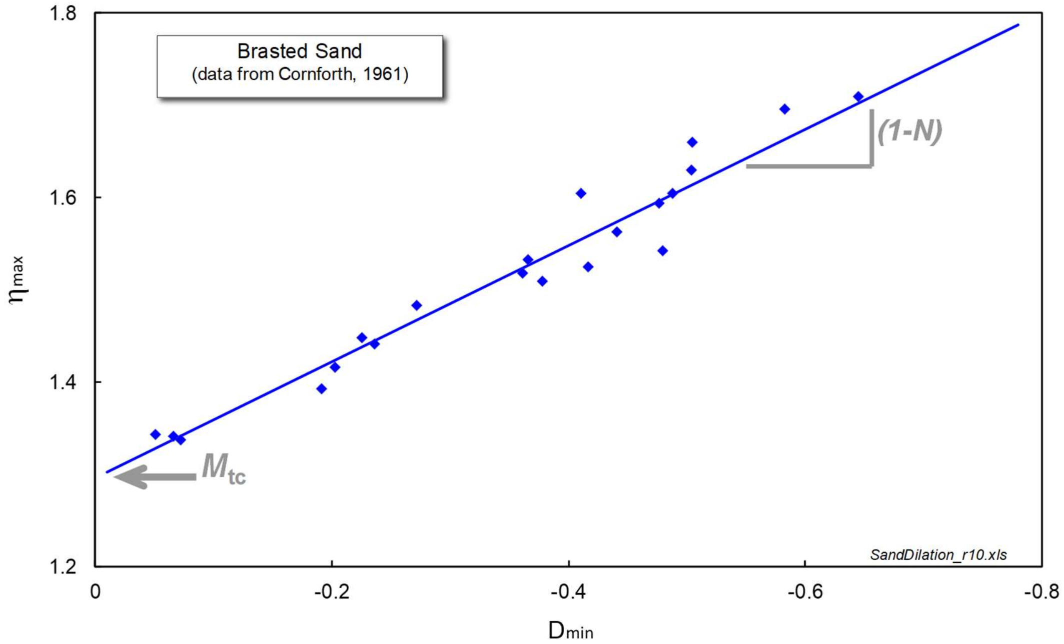


Figure 7: Drained strength of a sand in triaxial compression

Loose tests lie to the left of Figure 7 and show little dilation while those on the right are dense and dilate strongly. As expected from the Taylor-Bishop idea of two strength components a straight line is a good representation of the trends in this test data. Brasted sand is not unusual, and similar results to Figure 7 will be found with all other soils. However, there is a deviation from the Taylor-Bishop idea in that the trend line does not have a one-to-one slope, being fitted by the equation:

$$\eta_{max} = M_{tc} - (1 - N)\, D_{min} \quad [4]$$

Maximum dilation, in the sense of maximum volume expansion, is actually D_{min} because of the compression positive convention of soil mechanics. Hence η_{max} occurs at D_{min}. The terms M_{tc} and N in equation [4] are soil properties. As D=0 at M_{tc}, M_{tc} corresponds to the Taylor-Bishop internal friction; N represents the proportion of work going to volumetric, rather than distortional, strain. Equation [4] was first stated by Nova (1982).

The behaviour shown on Figure 7 was for a sand; there is nothing unusual about that sand and others sands show similar behaviour (although the properties M_{tc}, N differ from one sand to another). But this effect of dilation on strength has no knowledge of geology – it derives simply from considerations of work being done; thus we would expect it to apply to clays just as well as sands. Drained tests on clay take a long time, and are not routine in normal engineering. But drained tests on clay have been used for research, in particular during the 1950's at Imperial College; Figure 8 shows data from tests on Weald Clay (Parry, 1957). Exactly the same behaviour is seen on Figure 8 as Figure 7.

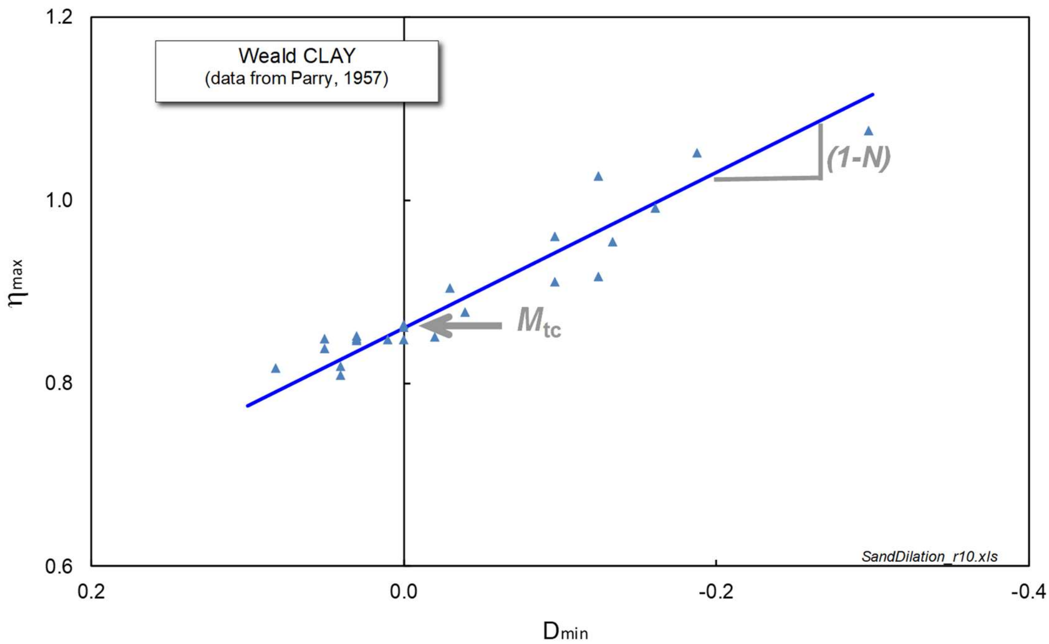


Figure 8: Drained strength of a clay in triaxial compression

Stress-Dilatancy

Taylor and Bishop's ideas on two-components to soil strength developed from the context of direct shear (shear-box) tests where the work components are obvious from the arrangement of the test. But there is more to this work-based idea than just strength. Looking back at Figure 5, it is obvious that the mobilized shear stress, throughout the entire stress-strain behaviour, is strongly influenced by the evolution of the rate of change of void ratio (dilatancy). This further development of the two-component idealization is easiest seen if we take test data and plot the mobilized stress ratio η versus the current dilatancy D. Figure 9 shows two drained triaxial compression tests, one on dense sand and one on loose sand. The measured data is shown on the left-hand side, and there is nothing unusual about these tests. The right-hand side shows the same data, but now transformed using equation [3]. As can be seen, what looks at first glance to be very different behaviour of the loose to the dense soil is actually a near common response of mobilized stress to dilation: *stress-dilatancy* as the behaviour is known. Really, this takes us back to Figure 2 – any deformation involves dilation, but it takes imposed stress to cause that deformation. The coupling between ratio of deviatoric stress to mean stress with dilatancy (= *rate* of dilation) is intrinsic to particulate materials, including soils whether gravels, sands, silts, or clays (or any mixture of these); it is a direct consequence of dilatancy being a work transfer mechanism, exactly as identified by Taylor-Bishop in their assessment of how strengths developed in a shear box.

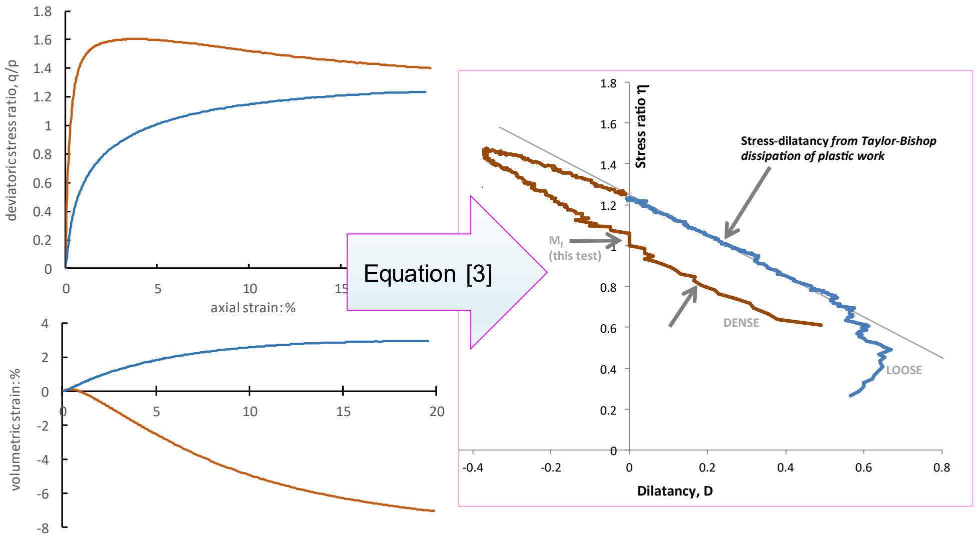


Figure 9: Drained triaxial compression data transformed to stress-dilatancy

Critical Void Ratio

It is apparent that the very loose packing of particles sketched on Figure 2 will contract when distorted while the alternative a very dense packing will expand – the change of volume with distortion will depend on the void ratio. Of course, soils comprise a distribution of particle sizes but that does not change the basic insight from this simple idealization. And, it is no great leap of the imagination to think that there may be a void ratio between these two limits where soil deforms at constant void ratio.

Hydraulic fill dams were a common construction method during the late 1800's to early 1900's, an era when there were pumps but not earth moving machinery that we have today; hydraulic filling was perceived as an inexpensive dam construction method. However, these hydraulic-fill dams had a propensity to suddenly fail by liquefaction slides during construction (much as is a current problem with tailings dams). It is obvious that if the fill is saturated (inevitable with hydraulic fills) then contractive response with cause positive excess pore pressure. This understanding was reflected in engineering the first liquefaction-resistant dam at Franklin Falls (New Hampshire) by the Corp of Engineers, directed by Lyman (1938), although more widely associated with Casagrande (1936).

The canonical result of the testing is shown on Figure 10 as the evolution of sand strength in a shear box with strain and its dependence on void ratio. A common end point was found, denoted as the critical void ratio which was the balance between some soil particles moving apart while others were falling into void space as the soil was deformed (in essence a dynamic situation of birth-death processes in terms of the contacts between the particles). And "critical" really meant what it said – it was the criterion of a safe density in constructed engineering work, with the practical concern to avoid sudden transitioning of drained construction with no excess pore pressure into an undrained liquefaction failure.

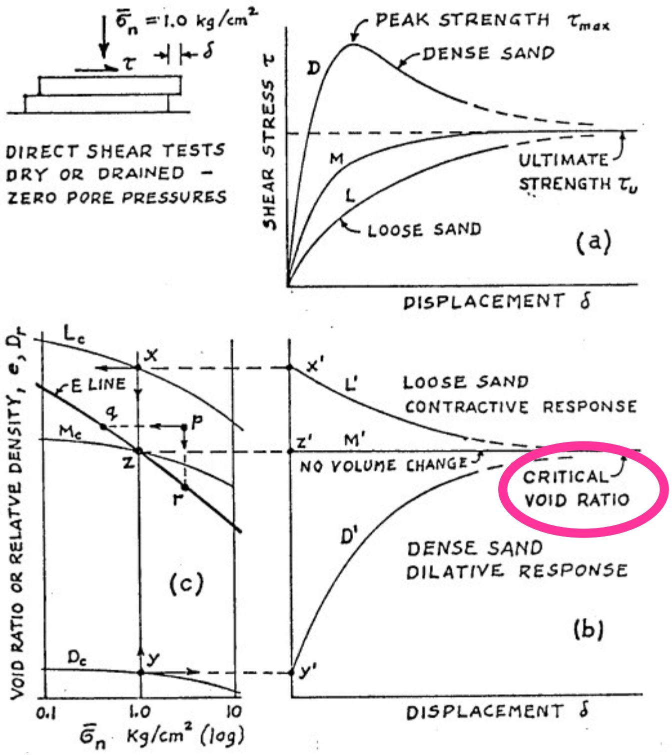


Figure 10: Origin and meaning of the critical void ratio, Casagrande (1936) (*author's emphasis*)

The critical state highlighted on Figure 10 comprises *two* conditions: void ratio change with strain is zero; and that absence of void ratio change continues indefinitely. These ideas are best expressed using dilatancy D as:

$$D = 0 \qquad [5a]$$

$$\delta D / \delta \varepsilon_q = 0 \qquad [5b]$$

If only condition [5a] is met, for example if you look closely at the volumetric strain behavior of the dense test shown on Figure 9 D=0 occurs at about 0.5% axial strain (when the volumetric strain trend reverses), the soil can be far from the critical state. D=0 can be, and usually is, a transient condition. In undrained tests, D=0 corresponds to where the stress-path reverses direction in those tests showing an ‘S’ shaped path, commonly called the ‘pseudo steady-state’ or ‘phase-change’ or ‘quasi steady-state’. Relying on [5b] is crucial to getting the critical state correct.

Triaxial testing is used to determine the critical void ratio in sands, silts, and clays. It is much easier to work with loose (contracting) samples because dense samples tend to form shear-bands (‘localize’) and where the void ratio of the overall sample does not reflect the void ratio in the intense shear zone. Both drained and undrained tests are used.

Undrained triaxial tests on loose soils often reach their critical state at less than 10% axial strain and well within the limits of the equipment; Figure 11 shows an example. Notice how the sample continues to deform at constant *p,q* in the critical state.

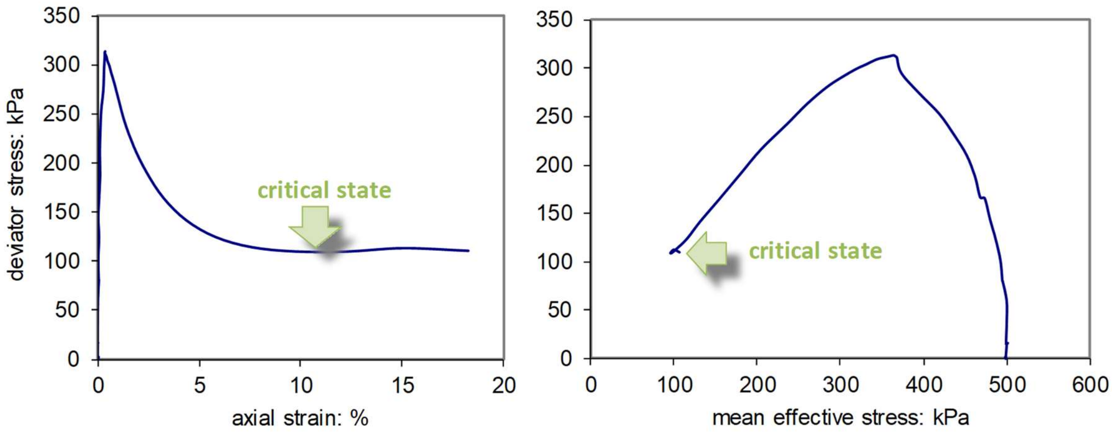


Figure 11: Example of critical state in load-controlled undrained triaxial compression

Casagrande appears to have viewed the critical void ratio (e_c) as a constant, akin to e_{min} and e_{max}; but that was quickly found to be wrong. Fort Peck dam failed shortly after the work by the Corps at Franklin Falls. Taylor (working at MIT, just down the road from Harvard in Cambridge, Massachusetts) applied Casagrande’s ideas about the critical void ratio to the investigation of Fort Peck and found that the critical void ratio depended on the effective confining stress (see Fig 14.12 of Taylor, 1948) – a dependence of critical void ratio on mean effective stress that today is known as the critical state locus (CSL). Taylor’s identification of the critical void ratio was not that used today because he focused on zero volumetric strain at peak strength rather than zero dilation rate, but this only causes an offset in the estimate of the CSL and does not negate the importance of identifying Casagrande’s error.

However, most investigation of this dependence on stress level was in the UK rather than the USA, and was for clays not sands; part of the reason for this change in emphasis is that void ratio (at least at that time) was a lot easier to measure with precision in clay samples than in sand. Various workers contributed to the testing, with Parry (1958) giving a very good synthesis and which supported a unique CSL, regardless of whether the tests were drained or undrained: Figure 12. Parry's results can be expressed as:

$$e_c = \Gamma - \lambda_{10} \log (p_c') \qquad [6]$$

…where e_c is the critical void ratio and Γ, λ_{10} are the soil properties (the subscript '10' denotes the logarithm base). Parry also observed that the CSL and isotropic normal compression line were parallel in the tested (remolded) clays; a corollary of which is that $\lambda_{10} = C_c$ where C_c is the familiar compression index.

This semi-log form of the CSL given by Equation [6] is usually adequate for engineering, in sands, silts and clays, over the stress range 30 kPa < p' < 800 kPa. Outside these limits, an upper void ratio limit may be needed on one hand while on the other recognition of constant compressibility may be appropriate at high confining stress; Verdugo (1992) provides insightful comment. Neither of these factors in any way affects the validity of the CSL as a limit condition in shear – rather, it is a matter of what level of detail is wanted in the representation of the CSL.

Leaping a little ahead of the historical story, accurate determination of the CSL in sands developed at Harvard in the late 1960's (Castro, 1969) and gained considerable ease and accuracy in the 1980's with the development of post-test sample freezing (in Calgary, see Been et al 1991). Nowadays, accurate CSL determination can be done by any good soil mechanics laboratory – although engineering judgment is needed in the data evaluation.

An example of a CSL in sand determined with modern methods is shown on Figure 13; the 'blue dots' on this figure are the identified critical state by inspection of the stress-strain curve in undrained tests while the loose drained tests do not quite get to the CSL within the limits of the equipment (and thus the CSL is inferred as a little lower void ratio than attained in these loose drained tests). Overall, the CSL shows a power-law, rather than semi-log, form; this kind of power-law seems most common for sands formed with a significant fraction of non-quartz particles. Grain-crushing may be a further contributor.

There has also been much confusion about 'uniqueness' of the CSL. This generally (invariably?) derives from experimental workers who fail to understand that the critical state is not a transient condition (commonly called the pseudo-critical state). More formally, uniqueness simply means that any combination of the three effective principal stresses (σ_1, σ_2, σ_3) produces a single value for the void ratio; physical sense further requires that increasing mean stress implies decreasing critical void ratio.

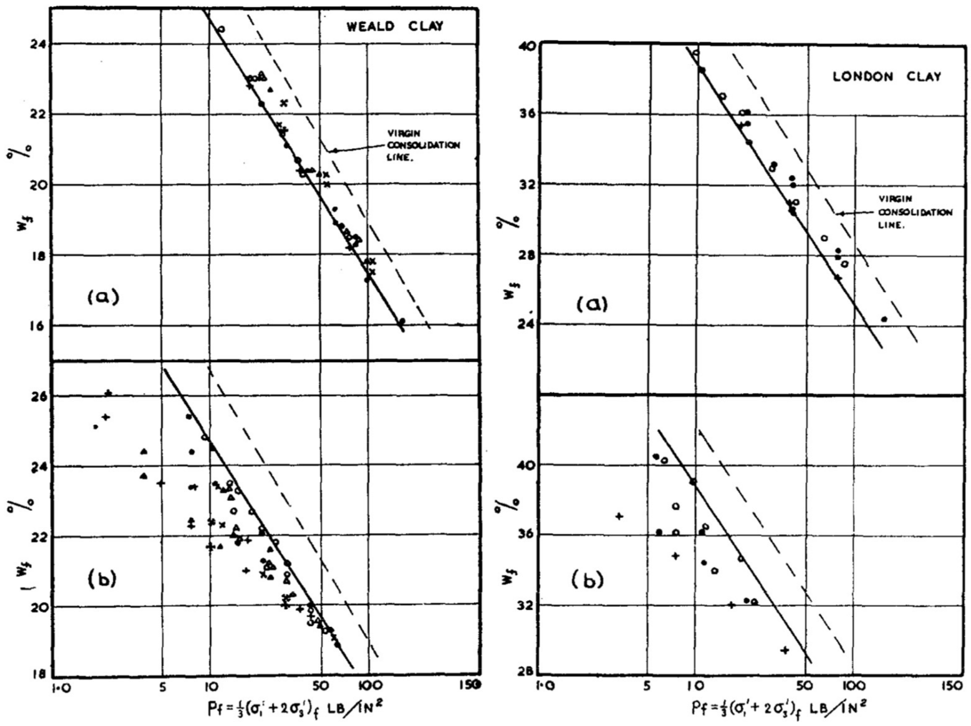


Figure 12: Rates of changes in dilation (drained tests) and pore pressure (undrained tests) at peak strength. Loose contractive samples are plotted in (a) and dense dilatant samples in (b). The bounding line in each pair of plots is common, and inferred to be the CSL. From Parry (1958).

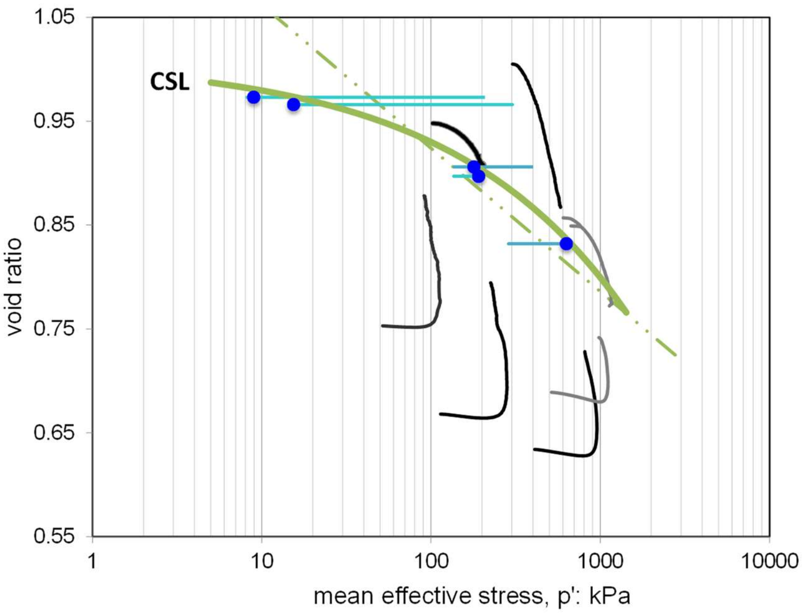


Figure 13: Example of a CSL in sand determined using modern methods (*note the departure of the CSL from the semi-log form shown as dashed line*)

State-Dilatancy

Taylor and Bishop's ideas on soil strength having two components provides an excellent understanding of soil behaviour but, looking at Figures 7 and 8, begs the question: why does a particular value of D_{min} develop? There is a missing component.

The starting point to progress is to look at Figure 13, and the various tests shown on it. Notice that the tests can start anywhere, both in terms of initial stress and initial void ratio; let us call the combination of *e,p* at starting point state 'A'. Wherever we start, we end up on the CSL (the green line); where on the green line will depend on the loading path, but let us call the end point *e,p* state 'B'. In physics, if you are going from state 'A' to state 'B' then you expect the speed of movement to be proportional to the distance from the end state. You will have encountered this idea in high-school physics when introduced to radioactive decay, and you will know the same idea in geotechnical engineering from a falling head hydraulic test (laboratory or insitu); in mathematical terms, both processes are described by the *first-order rate equation* (check your undergraduate math text…). Soil mechanical behaviour is no different.

A measure of deviation from the end state is needed. The end state is the critical void ratio and, to keep things simple, we define that deviation measure at constant mean stress:

$$\psi = e - e_c \qquad [7]$$

… where ψ is the soil's *state parameter*, the name reflecting that any soil exists over a spectrum of void ratios and we need a state measure analogous to temperature of a gas. Notice that the state parameter is not affected by the particular shape of the CSL or the equation used to describe that experimentally determined shape.

That the critical void ratio is the endpoint of all loading paths is then given by requiring that the deviation measure decreases with distortion:

$$d\psi = -g(\psi)\, d\varepsilon_q \qquad [8]$$

Physical sense requires that the function *g*() be 'single-valued' and return positive values for positive ψ, with the negative sign emphasizing that the deviation from the end state is to be reduced with distortional strain. A little bit of mathematics (see Appendix A) leads to the elegant *state-dilatancy* relation:

$$D_{min} = X\, \psi \qquad [9]$$

…where X is a scaling coefficient and is a property of the soil (in critical state soil mechanics it is a convention that all properties are upper-case Greek font; here the property is upper-case 'chi'). Putting equation [9] in [4] gives soil strength in terms of its void ratio and the confining stress.

Does the mathematics work? Figure 14 shows some test data, and it is immediately apparent that [9] is a very good representation of what was measured.

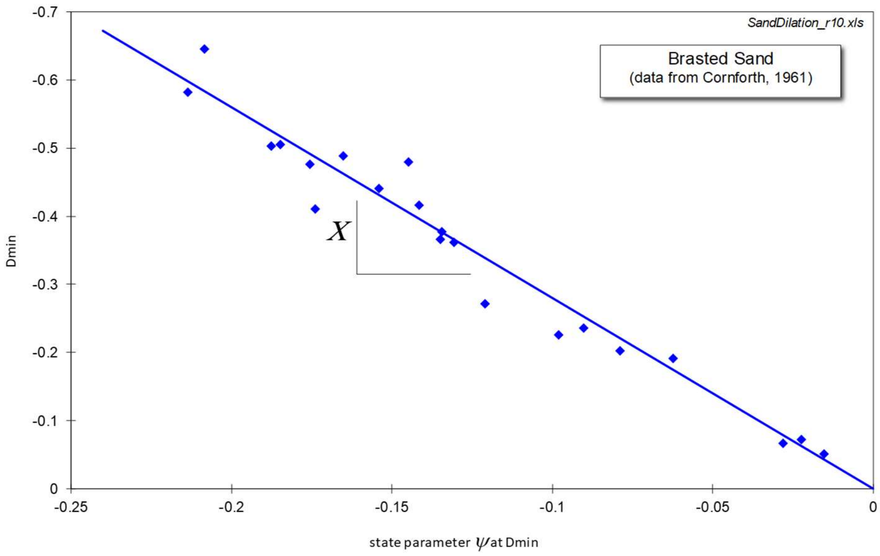


Figure 14: State-dilatancy of Brasted sand

One caution. The widely-cited Been & Jefferies (1985) was a 'laboratory paper' that synthesized a large body of triaxial test and, as such, it was natural to work with the sample void ratios at the start of loading – all very convenient, and requiring no assumptions or models. But, in terms of mechanics, actually not acceptable. Thus, Figure 14 is given in terms of the state parameter ψ at the instant of D_{min} – a bit of data processing is needed for proper mechanics.

Tutorial 1 – Determining soil properties

Using the supplied triaxial data on Nerlerk sand:

- Determine the CSL of Nerlerk sand using the undrained tests to give the soil properties Γ, λ_{10}
- Numerically differentiate (using *central difference* method) the reported data for Test 3 to compute D vs axial strain
- Find D_{min} and η_{max} for each test and plot these results to determine the properties M, N
- Using the determined CSL, determine the dilation coefficient X.

Elasticity

Elasticity is, possibly surprisingly, essential to understand how undrained strengths and stiffness develop. Because elasticity is common to everything that follows, it is helpful to present it now before dealing with plasticity. Elastic behavior used to be difficult to measure (requiring very precise transducers) but the advent of geophysical methods, both in the laboratory and insitu, has made elastic measurement routine in geotechnical practice.

Isotropic Moduli

Soil is generally anisotropic with, say, its stiffness in a horizontal loading direction being greater than for vertical loading. But, anisotropy is somewhat secondary to the basic influence of void ratio and there is the further point that there is no point in adding the complexity of anisotropy before you have an isotropic idealization working. There is also the important practical point that we have enough trouble measuring soil properties without adding the additional requirement that we must measure in every direction as well. Geotechnical engineering practice is dominated by isotropic elasticity and reasonably so.

You will have encountered isotropic elasticity as part of your engineering education, in particular with the two elastic constants of Young's Modulus (*E*) and Poisson's Ratio (ν). In the case of soils, it is very helpful to separate their behaviour into distortional and volumetric aspects; doing so makes it at least convenient to represent elasticity in terms of a shear modulus (*G*) and a bulk modulus (*K*). The relation between these alternative representations is:

$$G = E/\,(2(1+\nu)) \qquad [10a]$$

$$K = E/\,(3(1-2\nu)) \qquad [10b]$$

Elastic Strains

Because we are using work conjugate strain invariants, it is convenient to also have elastic strain increments in terms of these invariants. These are:

$$\Delta\varepsilon_v^e = \Delta p'/K \qquad [11a]$$

…and

$$\Delta\varepsilon_q^e = \Delta q/3G \qquad [11b]$$

Note the "3G" in [11b] which arises because of the definition of this invariant and which is not the "2G" you might have guessed from this being a shear strain measure.

Geophysical Measurement

The travel velocity of elastic waves is readily measured geophysically: the travel time of an identifiable signal is recorded between a source and a receiver and converted to a velocity using the length of the estimated 'ray path' for the elastic wave. The first-arrival is the ‘P’ (=compression) wave; the ‘S’ (= shear) wave travels more slowly, but is polarized – so ‘polarity reversal’ techniques are used to identify the S wave within other arrivals. Thus, any measurement of S-wave velocity involves two measurements. The modern technique is to make each of these ‘two’ measurements multiple times and ‘stack’ (= add) the signals, as that enhances the true signal over background noise.

The P-wave has limited value in soils as it is dominated by wave propagation through the pore water of the soil (if the soil is saturated, which is usually the case). But, the S-wave is now widely used because the S-wave velocity (V_s) is unaffected by soil saturation and directly related to the elastic shear modulus (G_{max}):

$$G_{max} = \rho \, V_s^{\,2} \qquad [12]$$

…where the ‘max’ subscript denotes that this is the limiting value for elastic shear modulus.

The geophysical techniques used in our industry include vertical seismic profiling (VSP) or enhanced CPT, both insitu, or ‘bender elements’ in the laboratory as an add-on to normal triaxial testing. All techniques are cheap, giving a lot of data for very few $.

Poisson’s Ratio

There is little data on Poisson’s Ratio because it is difficult to measure. If triaxial tests are used to assess soil elasticity, then ‘local strain’ transducers mounted on the sample are needed – which pretty much is rather fancy research testing. If geophysical methods are used, which are possible in principle because Poisson’s Ratio is a function of the ratio of the compression wave velocity in the soil skeleton and the shear wave velocity, then we run into the problem that the compression wave velocity in saturated soil is dominated by the compressibility of the pore fluid rather than that of the skeleton (practically, with a saturated soil, all you measure is the elastic wave velocity in water). Thus, there is not a great deal of research on the topic.

One of the most significant studies on the elasticity of soil was the extensive testing of Ticino sand by Bellotti et al (1996). Although this study was directed towards anisotropy, if the slightly anisotropic moduli of the soil are approximated with an isotropic idealization the data can be used to estimate Poisson’s Ratio in the various experiments. This approximation gives a plausible range of:

$$0.15 < \upsilon < 0.25 \qquad [13]$$

It is not uncommon to use this range for all particulate soils (ie those where the particles are somewhat spherical as opposed to, say, clay mineral platelets) because of the absence of other comparable studies. Practically, $\nu \sim 0.2$ is a plausible value for soil and consistent with soil behaviour in drained triaxial compression; it is our starting point.

Relationship Between G and K

Most (all?) soils appear to show a near constant Poisson's ratio (ν) even as the elastic moduli change with stress level, although it must be admitted that there is not a great deal of data. The problem is that measuring Poisson's Ratio in a triaxial test involves very fancy local strain measurements and with careful testing to ensure the sample is in its elastic zone. Or we can use bender elements, but then the sample has to be dry else we only measure the compression wave in water and which prevents calculating a correct elastic ratio. The canonical study on this topic is the extensive testing of Ticino sand by Italian researchers, see Bellotti et al (1996) in Geotechnique.

The apparent near-constant ν makes it very convenient to relate the bulk (K) and shear modulus of the soil (G) through:

$$K = 2G(1+\nu)/\,(3-6\nu) \qquad [14]$$

… and with the same dependence on stress level and void ratio between the two elastic moduli. Putting plausible values for ν in [14] gives:

K = 1.33 G … for ν = 0.2

K = 1.10 G … for ν = 0.15

That is, as a rule-of-thumb (given the variability in measured values), we expect broadly similar numbers for K, G regardless of any effect of density or confining stress.

Effect of Stress and Void Ratio on Elastic Moduli

Our starting idealization of soil as a collection of spherical particles immediately suggests starting with the 'classic' Hertz (1882) theory of deforming contacts between elastic spheres; this theory indicates that K should depend on confining pressure with an exponent of one third. Equally, there must be an effect of void ratio, as denser packings have more contacts. Some data, predominantly on silt tailings, illustrating these aspects is shown on Figure 15. Sands show comparable behavior.

Many workers have looked into how void ratio and stress level affects G, with various equations suggested to represent the two effects. An acceptable relation, and which works well across a range of soils, is:

$$G = \frac{A}{e-e_{min}}\left(\frac{p}{p_{ref}}\right)^{b} \qquad [15]$$

…where p_{ref} = 100 kPa by convention (this is done to avoid A having odd units). A, e_{min}, b are all soil properties: A is a normalized modulus in the same units as G (typically MPa); e_{min} amounts to the transition void ratio from 'particulate' to 'rock' and is somewhat denser than measured using the ASTM procedure (try e_{min} about 0.1 denser than that measured in a maximum density test); and, b is a modified form of the Hertz exponent, being typically about double the theoretical value of one-third.

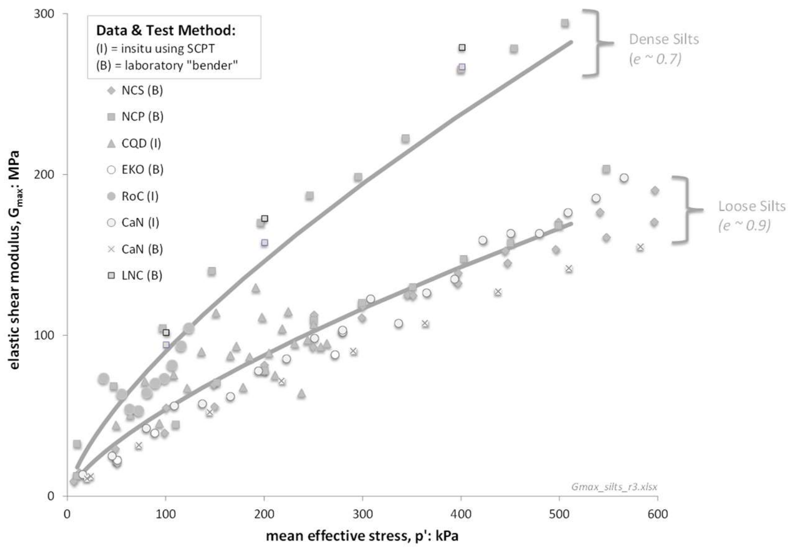


Figure 15: Examples of measured trends for G_{max} in silts

Strain-dependent Elastic Modulus

Applied mechanics does not allow for any strain-dependence of *elastic* moduli, and this very fundamental requirement is borne out by data. Figure 16 shows the results of a triaxial test on dense sand in which bender elements have been used to measure *G* at various strains: *G* does change slightly because of the increase and subsequent decrease in *p* during a standard drained triaxial compression test; this change in G is accurately predicted by [15].

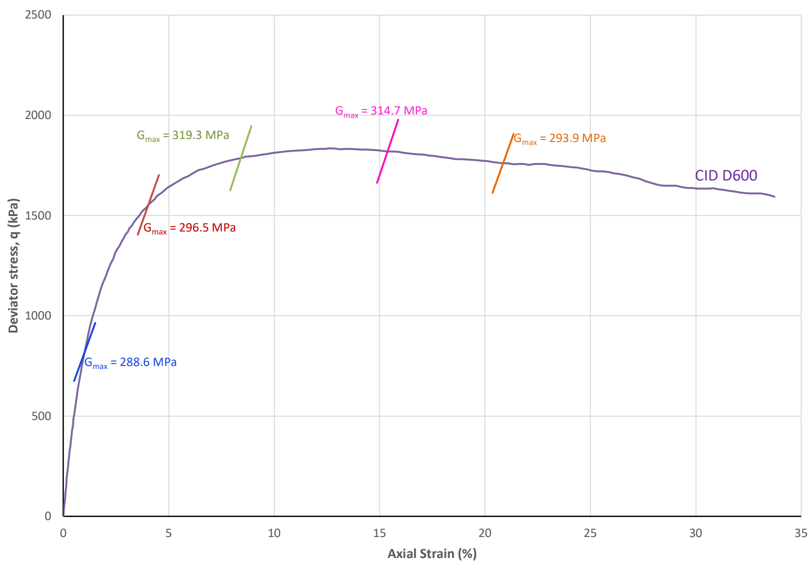


Figure 16: Measured G_{max} at various stages of a drained triaxial test on dense sand

Plasticity Theory

So far, everything has followed from the idea that the grains of soil you can see in your hand (or under a microscope) can be represented as essentially rigid particles that interact with each other as the soil mass is deformed. This idea has led to both stress-dilatancy (which gives detail about the stress-strain behaviour) and state-dilatancy (which gives the current strength). But there is still one important missing factor: soil deformations are irrecoverable most of the time. Understanding soil needs adding this irrecoverable aspect to the stress and state dilatancy: plasticity theory. There are always four elements to plasticity theory: yielding; plastic flow; work hardening; and the 'consistency condition'. These elements are discussed in this section.

Soil is Plastic

Undergraduate soil mechanics courses always involve some shear tests, possibly a simple shear box or perhaps a triaxial test. Loading tests on soil, whether clay or sand, will show that most deformation is not recovered when the load is removed. Figure 17 illustrates an example of a triaxial test, and defines the irrecoverable and recoverable components that are known as plastic and elastic deformation respectively. You can do your own tests easily enough and you will always find behavior similar to Figure 16; never doubt that plastic strains are an exceedingly important part of soil behavior.

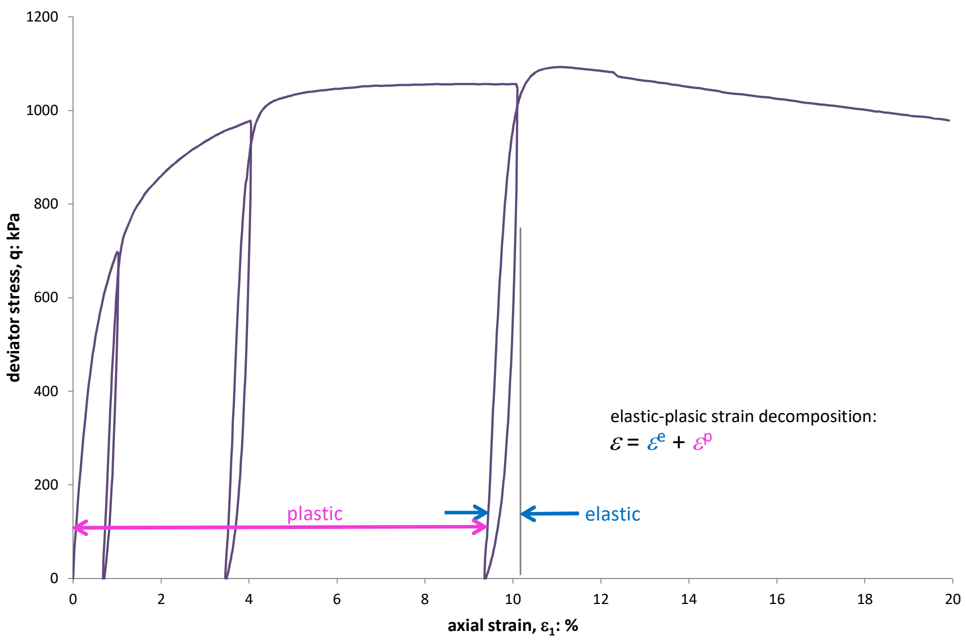


Figure 17: Occurrence and definition of plastic strain (dense Erksak sand, drained)

Plastic behaviour is not a recent subject, having been studied for some 150 years. Why so much study? Two basic reasons. First, because plastic behaviour is so common and, in the case of metals, the basis of many industrial processes (for example, designing the form for stamping the sheet metal to make your car). Second, because elasticity gives a one-for-one relation between stress and strain whereas plasticity is very different, plasticity often coming down to strain indicating stress but not the opposite – you cannot substitute plastic modulus for an elastic one; a new theory is needed.

The separation of displacements (and strains) into the two components illustrated on Figure 17 is fundamental to representing the mechanical behaviour of materials at the macro scale; this separation is known as the *elastic-plastic strain decomposition* and applies to any strain increment $d\varepsilon$ (i.e. be the strain principal, in a particular direction, or an invariant):

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \qquad [16]$$

…with the superscripts denoting elastic and plastic. Equation [16] is universal to plasticity theories whether for soil, steel, copper etc. Elastic strains are familiar, being the starting concept of every undergraduate civil or mechanical engineering course. Plastic strains are perhaps less familiar and connected to the idea of 'yield', the stress conditions that allow plastic strains to arise.

Yielding and Yield Surfaces

Coulomb (1733) identified yielding as an important behaviour in soil, but it was Tresca (1864) who carried out experiments on the punching and extrusion of metals and first formalized a general yield condition. Subsequently, various workers followed two routes although restricted to metals. Some investigated the yield condition with various experiments. Others invoked physical or mathematical ideas, for example the Von Mises yield criterion corresponds to a limiting value for the elastic shear strain energy.

Regardless of how derived/investigated, a yield surface is simply the limit of a region in 'stress space' within which all stress increments produce only an elastic response; you can choose your stress space to be expressed by principal stresses, but it is generally more useful to show (and express) the yield surface using stress invariants. The region of elastic response is defined by a continuous (and convex) curve: the *yield surface*.

Overall, by about 1935 there was a complete theory of plasticity for metals; in this context, what is meant by 'theory' is a complete framework (set of equations) that broadly matched experimental observations. This plasticity theory does not explain the underlying behaviour of the metal that causes plasticity, which is actually the movement of dislocations within the metal. Plasticity theory is an abstraction for representing the effect of underlying processes; it gains a fundamental slant through invoking the *Second Law of Thermodynamics* in considering plastic work (as plastic strains are permanent, work is done and dissipated as heat – bend a piece of metal repeatedly and feel it warm up to the touch).

Although Coulomb identified plasticity as intrinsic to soil behaviour nearly three hundred years ago, not much developed in terms of applying plasticity theory to soil until the 1950's. Then, much like metals, developments with soils included both experimental work and theoretical studies. However, both approaches were complicated by two aspects where soils differ from metals: soil yielding depends on the mean stress (which has no effect in metals at all) and soil generally changes volume during plastic distortion (while metals yield at constant volume). With the exception of the undrained strength of clay as represented in a total stress paradigm ("$\phi = 0$" and so forth), which reasonably matches Tresca plasticity if the soil is not brittle, something more than just adopting metal plasticity was needed – which should not be a surprise as soil plasticity is about realigning particle contacts and void space change rather than the movement of dislocations between crystals in metals.

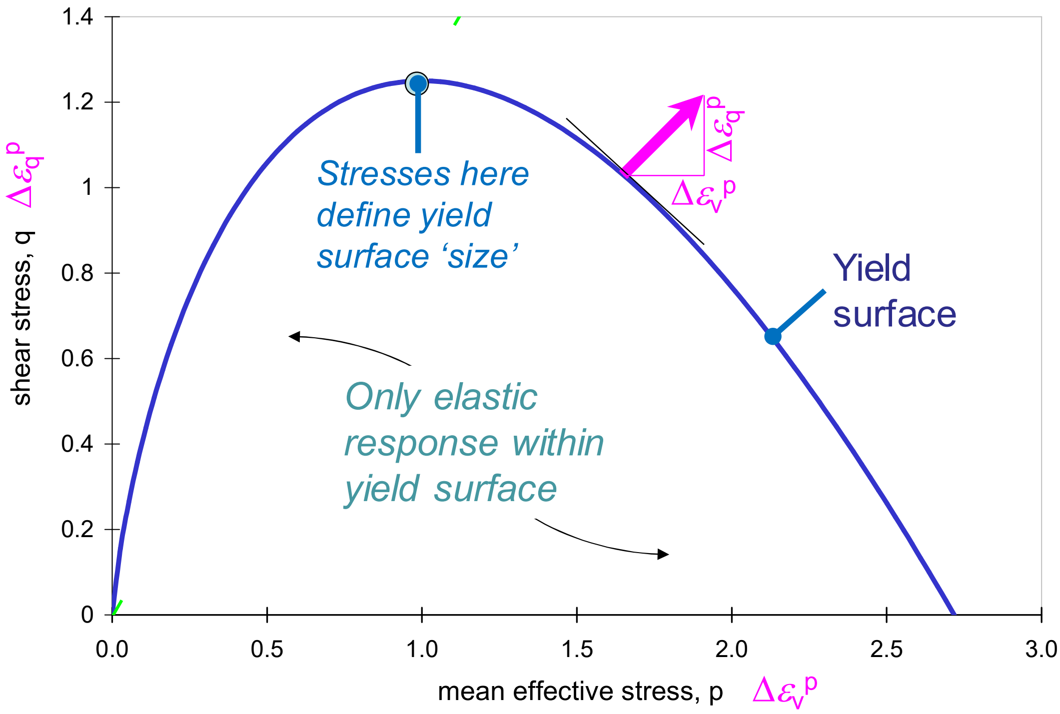


Figure 18: Example of a yield surface (shown using invariants to define the space)

Getting slightly ahead of things, Figure 18 shows an example of a yield surface for soil; we will derive this yield surface later in these notes. Yield surfaces are expressed by equations, the equations being in terms of the stress system adopted (principal, cartesian, or invariant) and one or more parameters characterizing the size and shape of the yield surface in terms of the chosen stress system – think of the yield surface as defining the 'strength' of the soil.

Classically, all the terms defining the yield surface are bundled into one side of an equation. Thus, the general form of the yield surface is:

$$F = f(\sigma_1, \sigma_2, \sigma_3, \text{'strength parameter'}, \text{'shape parameter'}, \ldots) \quad [17]$$

…where with this arrangement:

$$F < 0 \Rightarrow elastic \quad [17a]$$

$$F = 0 \Rightarrow plastic \quad [17b]$$

Strain increments within a yield surface are always purely elastic, but that begs the question: what happens on the yield surface? By definition, plastic strains develop on the yield surface, but in which direction and to what extent? There are two ideas here, which are known as, respectively, the *flowrule* and the *hardening law* (roughly equivalent to Poisson's ratio and Young's modulus in elastic theory).

Flowrule

Since plastic strains only develop on the yield surface, it is natural to indicate the relative amounts of those strains in a diagram that includes the yield surface. This is done by plotting both stresses and strain increments on the same axes – see Figure 18. You can use σ_1, $\delta\varepsilon_1$ etc but the combinations of stress and strain increment used must be work conjugate as explained earlier in these notes; Figure 18 uses the work conjugate invariants for a triaxial test.

In Figure 18, the strain increment vector defines dilatancy; the basic behavior of soil seen in experiments simply translates to a vector in stress space. At this point we run into an important idea (actually, a theorem; Drucker, 1954) which is that, for work conjugate stress and strain increment measures, the plastic strain increment vector will be perpendicular to the yield surface: *normality* (also called an *associated flow rule*). This is the concept shown on Figure 18.

At this point we know the relative amounts of plastic strain, but not their magnitude. We need to consider how the yield surface responds to these plastic strain increments.

Hardening

Plastic theory may seem a bit of a steep learning curve at this stage, but you will already be familiar with some aspects and what you are familiar with is now important. Think of the pre-consolidation pressure during oedometer compression of a clay sample, which pressure is no more than the yield stress for a particular loading path. And, as you accumulate plastic strain ('normal consolidation') then the pre-consolidation pressure increases: *plastic work hardening*. This behavior is generalized from the familiar oedometer test to the behavior of yield surfaces.

Yield surfaces generally evolve, expanding or shrinking with plastic strain. The way this works can be seen on the drained triaxial test shown earlier on Figure 17. Consider the first (left hand) unload-reload loop; clearly most strain is not recovered, so the stress-state at the start of unloading is on the yield surface. Now consider the reloading. Initially, everything is elastic until the yield surface is met (as per Figure 17) and at which point plastic strains restart. But, notice that the stress-strain curve then shows increasing strength with those plastic strains – this is the general form of hardening that you are familiar with in an oedometer but now carried across into 3D. What is now needed is to codify how yield surfaces change size with plastic strain: the *hardening law*.

The hardening law is as fundamental to plasticity as the yield surface. If the yield surface always has the same shape, and that is the simplest starting point, we can specify the yield surface size by a single parameter and there are two obvious choices to do this: the mean stress at maximum deviator (p_{qmax}) or the intersection of the yield surface with the mean stress axis (= the pre-consolidation pressure under isotropic compression). Which of the two options should be chosen ?

A tacit view in the ideas sketched out so far is that plasticity is about large deformations and with the concept of 'strength' in the background. Conversely, oedometer (or isotropic) compression is all about confined stiffness. So, it is no surprise that p_{qmax} became the preferred characterization of yield surface size for soils. And the linkage to the critical state is then immediate: because $p_{qmax} \Leftrightarrow D = 0$, state dilatancy – a core concept from our particulate idealization – becomes a natural control on yield surface evolution. More formally as $D = 0$ is one of the conditions for the critical state we want to make this link clear – so, we refer to this scaling stress as the 'image' condition as in a way it is like seeing an appearance of the critical state but we are generally not there yet; this image stress is denoted as p_{img}. The hardening law will then have the general form:

$$dp_{img} = h\left(\psi,\ \sigma_m, \sigma_q, p_{qmax}, properties\right) d\varepsilon^p \qquad [18]$$

…where $h()$ is the hardening function that will be specific to a particular idealization (or model). Notice the hardening law is written in terms of plastic strain, consistent with the theoretical framework. If you want a yield surface to change shape as well as size, then more measures are needed; for example, if the yield surface was taken as an elipse we would need to have a property specify the ratio of major to minor axes as well as the measure of overall size. In reality, we can represent a great deal of soil behaviour just using a simple (= constant shape) yield surface like that illustrated in Figure 17 and there is no reason to add unnecessary complexity.

If you move the plastic strain increment from the left-hand side of [18] to the right-hand side, then $h()$ can be viewed as a plastic stiffness or modulus. This is true of all work-hardening plasticity: the theory gives you a current stiffness, not a stress-strain curve. This stiffness has to be integrated in our stress analysis to get the overall behaviour we are interested in – which is not difficult, and we will come to that shortly after dealing with one more aspect of plasticity theory.

Consistency Condition

Work hardening (and softening) plastic models change the size of their yield surface with plastic strain. The *consistency condition* is simply that the stress state must remain on the yield surface during plastic strain, so that the stress state evolves on a one-for-one basis with the evolution of yield surface size. For the universal description of yielding as F=0, the consistency condition is just:

$$dF = 0 \qquad [19]$$

The consistency condition is illustrated in Figure 19, which shows an initial yield surface that has hardened after an increment of plastic strain. The question then is: how the stress state has evolved? We could try working out the intersection of the imposed loading path with the hardened yield surface – but, this is both tedious for an element test when we know the path and essentially impossible in general as it would involve solving simultaneous equations at every integration point. So, conventionally, [19] is exploited with the yield surface differentiated to provide a further equation that is applied incrementally. Don't worry, this is simpler than it sounds as will be seen shortly.

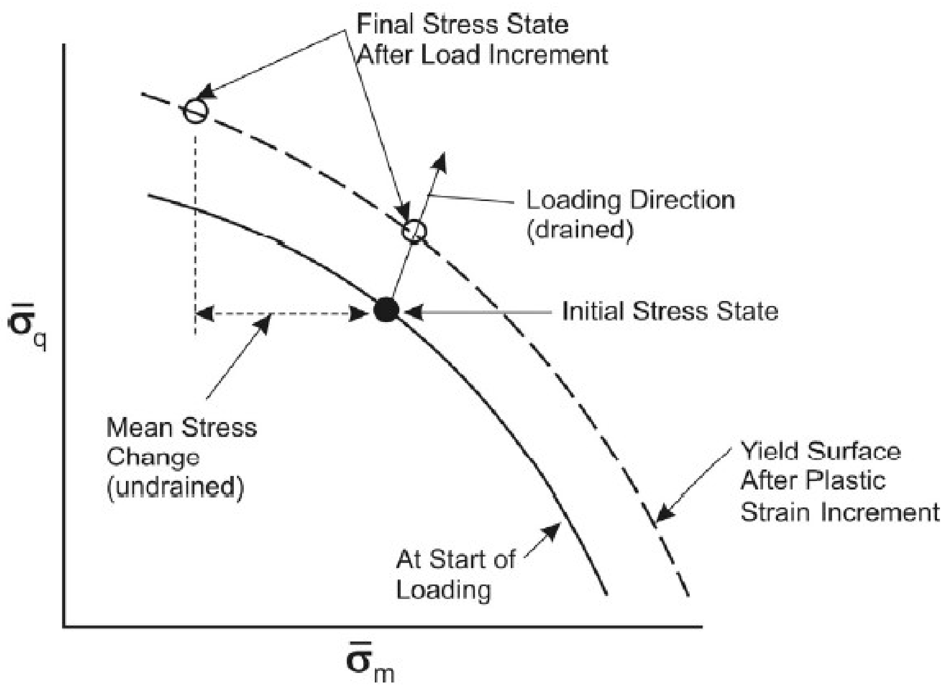


Figure 19: Illustration of the consistency condition
(*triaxial drained and undrained paths shown*)

A Numerical World

Some aspects of soil behavior lend themselves to being described by simple equations, but generally we need to work with numerical methods. This simply is not an issue as all engineers use spreadsheets today rather than calculation pads and sliderules that dominated our calculations fifty years ago – and your starting data, whether oedometer, triaxial or CPT – will be digital. But there is a little more to things than 'just use a spreadsheet'. First, we need to use spreadsheets properly. Second, we need to use numerical integration of the governing equations for soil behavior. Neither aspect is difficult, and the needed techniques are discussed in this section.

Visual Basic for Applications

Spreadsheets (and here we take Excel as the default program) are now widely used in geotechnical engineering and are ubiquitous for assembling and presenting the results of laboratory tests. There will be differences between companies as to what is the house style for graphs and the calculations, but most companies and academic faculty do a terrible job in their use of spreadsheets: in our experience (and we have reviewed a lot of spreadsheets) there widespread trend to put far too much on the worksheets with complicated formulae using cell references that are almost impossible to read and understand. It is like going back to the very early days of computing when we had to program an Apple II or Sinclair Spectrum and very simple names and expressions were all that were allowed. But, that is why computer scientists developed 'high-level' languages such as Fortran and which offer two key features for engineers: i) variables that can be written using plain-English and descriptive names (eg we can use '*lambda10*' rather than '*F21*'); and ii) arrays with the associated looping constructs.

Engineering has seen a range of high-level languages that changed (or rather become obsolete) rather quickly. Our problem as engineers is that our industry has minimal financial importance to the wider market that supports software development (just think of the vast intellectual effort going into 'apps' for your cellphone); a second problem is that the common expectation of everything having a graphical interface means the programming environment is changing as fast as new chips and operating systems evolve – yet all we want as engineers is the ability to calculate in a clear and structured way, and we would like to be isolated from changing fashions as to how things should look (which take time to fix when we would be rather thinking about our engineering). Those of us of a certain age just loved the language QB45 that came soon after the IBM-PC became the dominant computing environment in our offices; but QB45 died with Windows. However, Microsoft has not left us completely in the lurch and a wonderful language comes incorporated with their Office suite: Visual Basic for Applications (VBA).

VBA is a programming environment that works with the Microsoft Office applications, and here we are interested in Excel so we limit ourselves to that. Components (worksheets, graphs etc) are 'exposed' within VBA for the programmer to use and manipulate – almost anything you can do in a worksheet can be automated using VBA. As a rather simple language with a Fortran-like syntax (it actually looks close to QB45), VBA suits many geotechnical needs. You do not need the substance of a complete programming language to build professional-quality applications because VBA works within Excel. For example, you can calculate using VBA and then plot the results as a graph within Excel.

This is not a course on VBA and you won't need to program in VBA to learn CSSM. However, as you move from 'learning' to 'applying' CSSM you will need to know a bit about VBA. The 'learning' tutorials all start with standard worksheets as these allow the easiest understanding of how soil plasticity works – this is fine for the basic triaxial test with the simplest CSSM idealizations. But, when we start introducing evolving friction ratios or over-consolidation (for example) we need to add in the details/nuances of soil behaviour and that gets to be horrible in a worksheet while trivial in VBA (using functions). So, the public-domain software for real application of CSSM is built around VBA. You could ignore the VBA and just use the Excel applications provided but we hope that you do not – CSSM is not black-magic and if there is any one thing we hope you gain from this course it is just that appreciation. Knowing how to look inside VBA is key to you being confident.

The VBA 'integrated development environment' is included with Excel. You get to this by pressing the short-cut keys 'Alt'+'F11' or through the Visual Basic launch-point contained in the *Developer* tab that shows up in Excel, circled in Figure 20 below.

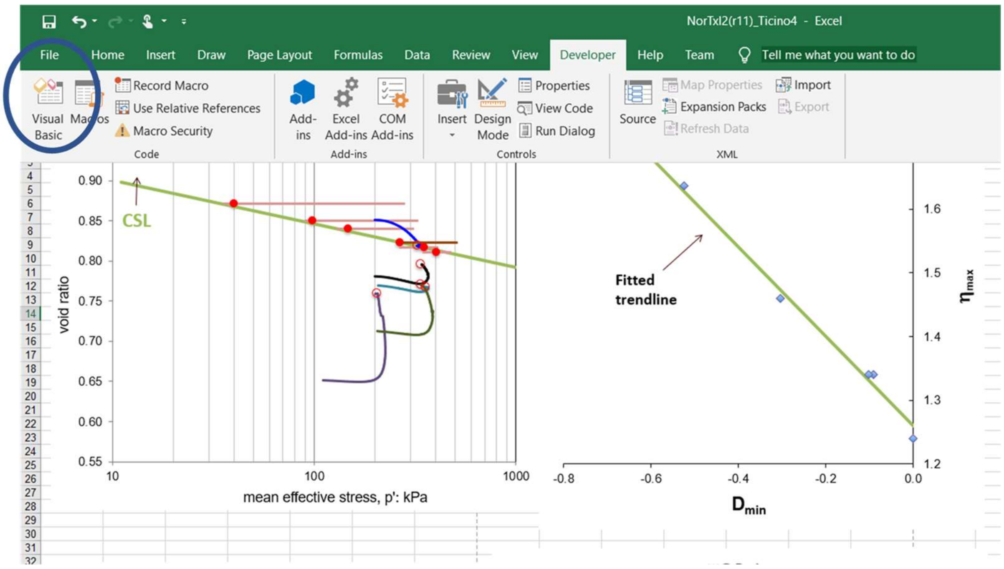


Figure 20: Accessing the VBA integrated development environment in Excel

There are books on programming in VBA, but what you need is really only to make yourself familiar with the provided code – and it is all written in 'plain English'. If you would like to know more about VBA, the book *Excel VBA Programming for the Absolute Beginner* by Birnbaum & Vine is good. You do not need it for the course, but it may be useful to you in the future as VBA is a truly wonderful feature within Excel for engineering. Despite saying this course is not about learning VBA there are three features that are really important in what follows.

The original intent behind the Basic language was that is should be simple and this led to not requiring 'declaration' of variables ahead of their use – which means if you make a typing mistake that mistake will not be detected (it will be treated as a new variable); and which leads to erroneous results and/or crashing applications. Get around this issue by forcing VBA into a professional mode by typing *Option Explicit* in the first line of the code. All the open-source code provided does this, and this option forces declaration of variables (which is also a good time to annotate what they do and their units).

Although VBA is integrated with Excel, there are different ways to do things. In our applications we will take some data, for example soil properties and initial conditions, and then compute the entire stress-strain behaviour as (say) 4000 points. We then use Excel to produce the various graphs. If each point is transferred back to the worksheet for plotting as it is computed everything becomes seriously slow and tedious. But, if the computation puts all the results into an array then that array can be transferred back to a worksheet very quickly indeed as a single item – which is the approach used. Thus, using CSSM (or processing CPT data) looks very much like an old-style Fortran program that operates when requested and dumps its results back into a worksheet for plotting. Easy to check or understand and quick to use. This operation is 'hung off' buttons on the worksheets; check the 'macro' (which is actually a subroutine) associated with the button (left-click if you are left-handed to get to the 'assign macro dialog) and then jump to the VBA code to see that sub-routine and follow where the code goes.

The final important feature is an item of programming style: 'named constants'. In many places we have options – drained or undrained, semi-log CSL or something else. We want the VBA code to be easily read in plain English and you cannot achieve that if these options are just referred to as numbers (1, 2 , 3 etc). You can use text strings to show these choices but it is far more efficient and elegant to declare constants as, for example, in this code fragment:

```
' CSL Mode
Const semi_log = False
Const curved = True
Dim CSL_choice As Boolean
```

…which allows *CSL_choice* to be used in code with its values shown as 'semi-log' or 'curved'. If there are more than two choices, just list them as constants but now the choice becomes declared as an integer rather than a true/false choice. Easy to read and understand, and widely used in software engineering.

Euler Integration

When the phrase “constitutive model” is spoken, people tend to think of integrals and fancy mathematics. That is a mistaken view, as very few realistic constitutive models for soil can be directly integrated to give a stress-strain curve for a laboratory test never mind used in analysis of engineering construction. Rather, what the constitutive models give is the equivalent of a current stiffness that evolves – and which can only be used numerically. Numerical integration is not necessarily difficult.

In the case of standard laboratory tests, if all is working ‘as advertised’, these tests all have the sample under uniform stress conditions – no redistribution of stress develops within the sample. In reality there are end effects (restraints) and strain localization (commonly later in a test), but if we treat the test as uniform conditions then standard laboratory tests have fixed stress or strain paths that allows simple numerical integration. The easiest numerical integration method to understand was introduced by Euler (1707-1783), a mathematician perhaps without peer.

Euler’s method is shown on Figure 21. Starting at point ‘j’, we compute the current gradient of the equation being integrated – shown as the tangent line. We then apply an increment along the path to move to the new position ‘j+1’, which in our case will be an increment of strain. Multiplying the gradient by the step length gives the change in value between two positions, and thus the new value of the equation at ‘j+1’. This method can be used in a spreadsheet, and that is what this course uses.

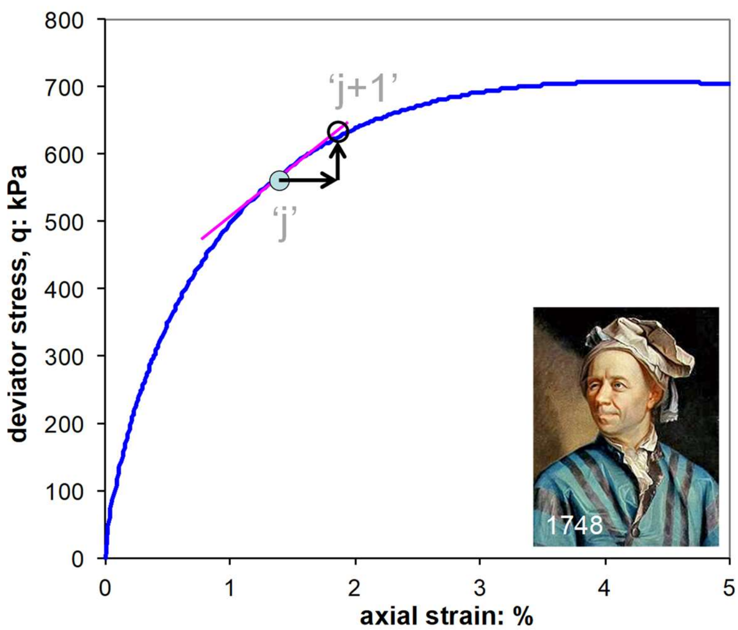


Figure 21: Principle of Euler’s method for numerical integration

The deficiency of Euler's method develops if the gradient changes quickly compared to the step size – look closely at Figure 21 and you will see the end point (open circle on the tangent line) is not quite on the curve we are trying to follow. It is easy to see a problem developing if the gradient at the end of a step is much different from that at the start; commonly, this shows up as the computed results "drift" away from the true solution.

There are more sophisticated numerical integration methods (eg Runge-Kutta), but they always involve more calculations. More sophisticated methods also cloud understanding of what is going on, a particularly undesirable aspect here as this course is about understanding how and why soil behaves as it does: a simple integration method is better even if lots of small steps are needed (and it turns out something like 3000 steps is plenty for a triaxial test, which is easy enough to implement in a spreadsheet). This course thus depends on Euler integration.

When you move from laboratory tests to stress-analysis of the ground in civil engineering works ('boundary value problems') an entirely different set of numerical procedures are needed – but, the derivation of the algorithms for finite element analysis is a specialist discipline and not something most geotechnical engineers do for themselves. At the time of writing, modern CSSM methods are available as public-domain 'user defined models' for some of the standard geotechnical modelling software suites and with a trend for their direct incorporation as 'standard' models in the software. However, do be careful as Modified Cam Clay is nearly universally offered by all software suites – and it is less than useless for most geotechnical engineering; you need *Bounding Surface* or *NorSand* variants based on ψ to get sensible representation of soil behaviour. But that is getting ahead of ourselves, as the first task is to understand how and why CSSM works.

Original Cam Clay

Original Cam Clay (OCC; 'Original' to distinguish it from 'Modified') was the first complete model for soil strength and stiffness. There were several preceding papers but these were in the nature of setting out the ideas and how they were developing; it is the Schofield & Wroth (1968) book that defines the framework and the OCC model. A feature of this book was its emphasis on linking void ratio to soil behavior, but this emphasis moved the OCC equations away from a standard plasticity view (which is actually simpler) and arguably is why people find OCC difficult to understand. Here we encapsulate the key ideas of OCC and present them for use in a worksheet for drained and undrained triaxial compression. We derive the model using both the 'void ratio' and 'plasticity' view, with the plasticity view being used for developing the worksheet.

Key Ideas

Some frameworks for understanding soil look to the experimental literature for their justification. OCC is very different. A much under-appreciated aspect of OCC is that it starts by adopting a few key ideas (assumptions) and then follows those ideas using formal mathematics; test data is only introduced at the end to see if the framework is relevant to engineering ('validation'). Of course, the ideas adopted were influenced by test data but the power in this approach is you can't reject OCC because you don't like the shape of the yield surface or the like. There are actually only three key ideas in OCC:

- The work dissipation assumption, which gives the stress-dilatancy rule
- Normality, which is used to derive the yield surface from stress-dilatancy
- All yield surfaces intersect the CSL, which gives the hardening rule

We will now look at these ideas in turn, seeing how they match our understanding of soil behavior, before implementing OCC in a spreadsheet.

OCC Flowrule

The starting point is the idealization about how plastic work is dissipated, and that idealization follows directly from the two-component strength model of Taylor-Bishop that was discussed earlier. If you turn back to Figure 3, the idea of dilation being a work transfer mechanism between the principal stress directions naturally leads to the idea that only the 'constant volume' friction dissipates work. Since only plastic strains do work (the elastic ones just store energy), the obvious step is to modify the stress-dilatancy rule that was inspired by observing the particulate nature of soils so that the earlier Eqn [3] now becomes:

$$D^p = M - \eta \qquad [20]$$

…where M is taken as a constant and the superscript 'p' denotes plastic. If you look at Bishop's work from seventy years ago (Figure 3), it is apparent that M evolves a bit at small deviator stress but M being invariant with strain, and independent of void ratio, is a pretty reasonable starting point. That is the first idealization of OCC.

OCC Yield Surface

The yield surface is derived from the normality condition. Differentiating $q = \eta\, p$ (which is the definition of η rearranged) gives:

$$dq = p\, d\eta + \eta\, dp \qquad [21]$$

Now think back to high-school math. If you have a line with gradient 'x' the gradient of a line perpendicular to that has gradient '-1/x'. The term dq/dp is simply the tangent to the yield surface; the line perpendicular to that tangent to the yield surface is the plastic strain increment vector from the normality condition (see Figure 17). Thus:

$$\frac{dq}{dp} = -\frac{d\varepsilon_v^p}{d\varepsilon_q^p} = -D^p \quad \text{or, re-arranged as} \quad dq = -D^p\, dp \qquad [22]$$

On substituting the stress-dilatancy rule [20] into [22] we get:

$$dq = -(M - \eta)\, dp \qquad [23]$$

And finally combining [23] and [21] to eliminate dq:

$$p\, d\eta + \eta\, dp = -(M - \eta)\, dp \;\Rightarrow\; d\eta = -M\, dp/p \qquad [24]$$

Equation [24] is a separated differential equation and solved simply by integrating each term, which gives:

$$\eta/M = -\ln(p) + C \qquad [25]$$

…where C is an integration constant. This integration constant is chosen as the mean stress at the stress ratio $\eta = M$, which is the stress-ratio of the critical state, and thus:

$$C = 1 + \ln(p_c) \qquad [26]$$

So finally we get:

$$\eta\, /\, M = 1 - \ln(p/p_c) \qquad [27]$$

The yield surface given by [27] is exactly that shown in Figure 17; the scaling stresses indicated on that figure are exactly p_c and $q = M\, p_c$. And notice that we have not used any test data in deriving the yield surface; rather, OCC is based on plausible physical equations inspired by observations about the particulate nature of soil.

OCC Hardening

The really important contribution of OCC was not the derivation of yield surface shape from simple physical principles but rather the linking of yield surface size to the soils void ratio. This provides the hardening rule and the linking is done via the CSL.

OCC idealizes the CSL using the widely-used, and quite reasonable, semi-log idealization:

$$e_c = \Gamma - \lambda_{10} \log(p_c) \qquad [6, repeated]$$

…where Γ, λ_{10} are soil properties that apply equally well to sands as clays (and despite the word 'clay' in OCC). There are two ways of incorporating [6] into OCC, which we will name the 'state view' and the 'plasticity view'.

Hardening via State

The Schofield & Wroth approach, widely shared by other workers at Cambridge, UK, was to emphasize the role of void ratio in soil behavior; in such a world-view the kernel framework is a void ratio versus log(p) plot, commonly called a 'state diagram' as it is a picture of where the soil sits in its void ratio space. Strictly, such state diagrams ought to include e_{min} but most workers just show the normal compression locus as this framework is much conditioned by experience with the oedometer test. Thus, [6] was rearranged, with a change from base-10 to natural logs (as appear in [27]) to give:

$$\ln(p_c) = (\Gamma - e_c) / \lambda \qquad [28]$$

However, there is a further step as OCC includes volumetric elasticity which means that *e* does not generally correspond to e_c – you have to allows for elastic void ratio change between the current stress *p* and p_c; OCC has the idealization in terms of soil state that:

$$e_c = e + \Delta e^e \qquad [29]$$

… where *e* is the current void ratio of the soil at the current stress *p*. The elastic behaviour is idealized from unload-reload behaviour seen in oedometer tests by introducing a new soil property κ (which is the oedometer property C_s but expressed in terms of natural logs) such that:

$$\Delta e = -\kappa \ln(p_c / p) \quad \textit{during unloading or reloading} \qquad [30]$$

…where the '–' sign arises because void ratio reduction is positive strain. Substituting [30] in [29] and the result in [28] gives:

$$(\lambda - \kappa) \ln(p_c) = \Gamma - e - \kappa \ln(p) \qquad [31]$$

Or on pulling everything together as a single equation showing how yielding and strength evolution depends on void ratio:

$$\eta / M = 1 - \ln(p) + (\Gamma - e - \kappa \ln(p)) / (\lambda - \kappa) \qquad [32]$$

Hardening via Plastic Strain Increment

Equation [32] directly links soil strength and stiffness to void ratio, but it is hardly intuitive. Worse, it does not look like a stiffness so it cannot be used in the finite element method in this form. And, it obscures the role of plastic strain – a role that is the kernel of the theory on which OCC is based. So, let us start with plastic strain increments.

OCC is inspired by an idealized oedometer test, so let us think like that and consider void ratio change as stress changes, Figure 22. Starting at stress state p_A and with matching deviator stress (q_A = M p_A) so the stress state lies on the CSL, let us impose sufficient deviator stress as we increase p such that the soil consolidates along the CSL to stress state p_B. The confining stress is then reduced, back to p_A, along with reducing the deviator stress, such that the soil unloads (swells) – some of the 'normal' consolidation is recovered during this unloading. If we view this behaviour incrementally the plastic (irrecoverable) void ratio decrease for loading along the CSL is:

$$de^p = -(\lambda - \kappa)\ dp_c/p_c \qquad [33]$$

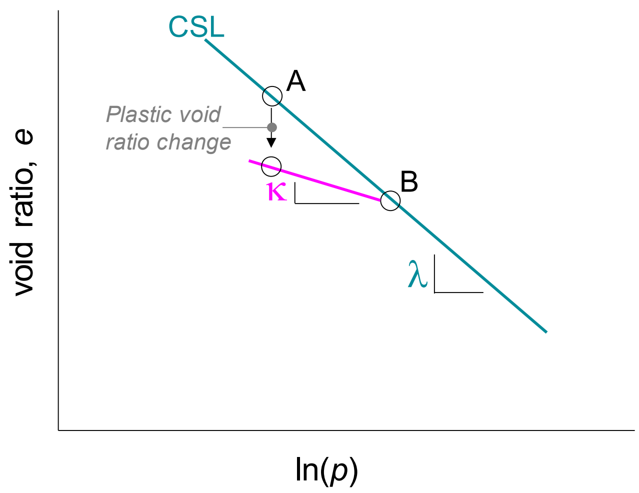


Figure 22: Loading and unloading in state space illustrating plastic void ratio change

We want to work with incremental strains rather than void ratio change, so noting that $de = -(1+e)\, d\varepsilon_v$ we substitute this in [33] and rewrite as:

$$(1+e)\ d\varepsilon_v^p = (\lambda - \kappa)\ dp_c/p_c \qquad [34]$$

There is now one further step. Recall Casagrande's original discovery of the critical state was about large-strain strength *in shear* (see Figure 10). Thus, we should work with $d\varepsilon_q$ rather than $d\varepsilon_v$ – which is easy enough to since by definition $d\varepsilon_v^p = D^p\, d\varepsilon_q^p$. So, using this identity we rearrange [34] as:

$$\frac{dp_c}{p_c} = \frac{1+e}{\lambda-\kappa}\ D^p d\varepsilon_q^p \qquad [35]$$

Equation [35] operates directly on the yield surface (equation [27]) to change its size in response to plastic deviatoric strain – easy to appreciate, and an elegant implementation of the Theory of Plasticity. This elegance has come at the cost that the direct linking to void ratio is slightly lost, although that linking survives in setting the initial value of p_c before applying plastic strain; equation [31] rearranged gives us the starting point:

$$\ln(p_c) = (\Gamma - e - \kappa \ln(p))/(\lambda - \kappa) \qquad [31a]$$

We also gain further insight as the parameter grouping:

$$H = \frac{1+e}{\lambda - \kappa} \qquad [36]$$

…can now be seen as a dimensionless plastic modulus (= soil property) analogous to the elastic shear rigidity I_r (= G/p). The slope of the CSL is not just about the critical state in itself, but rather reflects the overall stiffness of the soil in distortion.

Computing Triaxial compression of OCC

All work-hardening plastic models change the size of the yield surface in response to plastic strains (that is what their *hardening law* does), except when we get to the critical state where everything continues indefinitely at constant conditions. So what happened to our stresses as the yield surface changes size? The *consistency condition* is that the stress state must remain on the yield surface during yielding; this is true of every plasticity model, not just CSSM. In essence, the yield surface drags the stress state with it as the yield surface changes size. Practically, the consistency condition gives us a further equation and produces the slightly paradoxical result that while we discuss yield surface shapes when developing the theory such yield surfaces are rarely used. Rather, we take the differential of the yield surface to work out where an increment of plastic yield has taken us; working out the corresponding stress change from the consistency condition is a key step in implementing any constitutive model.

In the case of unknown stress paths, such as general analysis with the finite element method, the consistency condition is used to drive the solution algorithm. In the case of laboratory tests, where the equipment controls the stress or strain path, the consistency condition is used directly to compute the stress-strain curve. For the 'bullet shaped' yield surfaces of OCC, differentiating the yield surface (equation [27]) gives:

$$d\eta = M\left(\frac{dp_c}{p_c} - \frac{dp}{p}\right) \qquad [37]$$

…where the first term in the () on the right-hand side is given by the hardening law (equation [35] above). The second term in the () is test-specific and is what you have to work out; the steps to do this are as follows.

Undrained Loading

The undrained condition is ε_V=0 (= constant volume). Using the elastic-plastic strain decomposition, this condition can be expressed in terms of incremental volumetric strains, broken down into their elastic and plastic components, as:

$$d\varepsilon_v^e = -d\varepsilon_v^p \qquad [38]$$

On substituting the elastic soil model in [38] we immediately get the change in mean stress:

$$dp' = K\, d\varepsilon_v^e = -K\, d\varepsilon_v^p \qquad [39]$$

You can see from the above equation [39] why elasticity is so important to undrained loading even though you are using a plasticity model – the excess pore pressure simply reflects the balance between elastic and plastic processes, with each process being equally important.

Most importantly, notice that Poisson's Ratio does not appear in [39] and in particular we never, ever, use ν = 0.5 to get an undrained response; rather, everything follows exactly from the elastic-plastic strain decomposition as expressed in [38]. Further, in the case of OCC, the concept of Poisson's Ratio does not even exist – elasticity is only represented by the oedometer unload-reload idealization of the "kappa" model with:

$$K = (1+e)\, p/\kappa \qquad [40]$$

Computing the new stress state after an increment of plastic strain is now simple. The above undrained condition gives us the change in mean stress which we simply substitute into the consistency condition [37] (and including the plastic hardening) to get $d\eta$. We then apply Euler integration to get:

$$\eta_{NEW} = \eta_{OLD} + d\eta \qquad [41]$$

…and then on to the next step after adding in elastic shear strain.

Drained Loading

In the case of drained triaxial tests the slope of the load path $\Delta q/\Delta p$ = 3 (assuming that you are using standard equipment and not trying to get a particular stress path...) because of the arrangement of the test. We use this loading direction by first differentiating η:

$$\eta = q/p' \Leftrightarrow d\eta = \frac{p\, dq - q\, dp}{p\, p} = \frac{dp}{p}\left(\frac{dq}{dp} - \eta\right)$$

$$\Rightarrow \quad \frac{dp}{p} = d\eta\, /(3 - \eta) \qquad [42]$$

Substituting the consistency condition [37] in [42] to eliminate $d\eta$ gives:

$$\frac{dp}{p} = M\frac{dp_c}{p_c}/(3 + M - \eta) \qquad [43]$$

…which now allows is to advance the solution using [41] just as in the case of undrained loading. We just modify one column in the undrained spreadsheet.

Summary of OCC equations

Yield surface … or as *Consistency*	$\eta = M\left[1 - ln\left(\frac{p_c}{p}\right)\right]$ $\Delta\eta = M\left(\frac{\Delta p_c}{p_c} - \frac{\Delta p}{p}\right)$
Flowrule	$D^p = M - \eta$
Hardening	$\Delta p_c = p_c\ H\ D^p\ \Delta\varepsilon_q^p$ …where $H = \frac{1+e}{\lambda-\kappa}$
Elasticity	$K = (1+e)\ p/\kappa$ G = '*a very large number*'

TUTORIAL 2 – Implement OCC for a CIU test and *verify*

Use the supplied template *OCC_CIU_Ex2.xls* to calculate the undrained triaxial behaviour of OCC. This template is configured to model a CIU test from *q=0*.

Even though OCC does not include any concept of elastic shear modulus, add this in as you develop the xls as it will be needed later as the xls is evolved to more realistic representation of soil behaviour in later tutorials; just set G_{max} = 2000 MPa for now so computed elastic strains are negligible in OCC consistent with its assumptions.

Verify your numerical implementation by comparison to the closed-form solution given by Schofield & Wroth (supplied in the template and plots as the 'red dots'). The closed-form solution uses the same properties as your numerical implementation, so the two soultions change together if you choose different soil properties.

Investigate the dependence of numerical accuracy on numerical step-size. What would be a reasonable 'rule of thumb' for the needed step-size ?

TUTORIAL 3 – Implement OCC for a CID test and *validate*

Exercise 3a

Save a copy of the Exercise 2 xls with a new name *OCC_CID_Ex3a.xls*

Delete the 'red dots' from the plots (i.e. the closed-form solution)

Add and align a new plot beneath the q - ε_a plot: ε_a - ε_v

Modify your undrained OCC worksheet to compute drained triaxial behaviour

Calibrate drained OCC to the supplied data in the template CID-682; what values of M and λ_{10} give the best-fit ? What do you think of the fit ?

Exercise 3b

Save a copy of the Exercise 3a with a new name *OCC_CID_Ex3b.xls*

Change the plotted data to test CID-667

On the 'properties' sheet, honour the tests reported initial void ratio

Why does OCC predict very large strengths ?

NorSand

Original Cam Clay is mathematically elegant but only matches reality for loose soils. This begs the question: what has been missed ? It turns out, as will have been discovered in the tutorial exercises, that the assumption of yield surfaces always intersecting the CSL is generally wrong; rather, the CSL is the end-point and yield surfaces evolve with distortional strain to that end point. Thus, while many aspects of OCC can be retained its hardening law must be generalized. That generalization is NorSand (NS) and the generalization uses the state parameter. NS was derived out of experience with large-scale hydraulic filling in the 1980's and has evolved somewhat since first published in 1993. Partly, this evolution is because there are choices within an otherwise derivable framework. Partly, this evolution is because additional features have been identified that both add detail while simplifying NS. The presentation here of NS is that as of late 2018 although limited to triaxial compression for this course; the emphasis for the course is illustrating how little needs changing in OCC to get to NS that then works well for all soils.

Key Ideas

The ideas of OCC continue into NorSand (NS) but are further formalized as two axioms:

AXIOM 1: A unique CSL exists. This need not be the familiar semi-log form, with whatever equation fits the soil test data being acceptable (ie the CSL shape is viewed as a modelling 'detail' for any particular soil).

AXIOM 2: Soil state is characterized by the state parameter ψ with the axiom being that ψ reduces with increasing deviatoric strain; this is simply a mathematical statement of Casagrande's canonical plot (Figure 5).

With these axioms established, NS further adopts:

- The Taylor-Bishop work dissipation idealization, but in the modified form suggested by Nova that best-fits test data; this admits the possibility of plastic work in volumetric strains, not just deviatoric strains.
- Normality, which is used to derive the yield surface from stress-dilatancy; this is exactly the same as used in OCC.
- A hardening limit from the state parameter principle (equation [9], see Figure 14)
- A hardening rule derived from Axiom 2 (as opposed to just assuming the CSL is the hardening rule)

We will now discuss these ideas in turn, seeing how they change our understanding from OCC, before verifying that NS captures everything from very dilating dense soils through to static liquefaction of loose soil. Perhaps the kernel idea is that, while OCC only deals with plastic strain increments, NS requires continued reference to an evolving ψ, so, your worksheet will need an extra column for the state parameter ψ.

NS Flowrule

The starting point is the idealization about how plastic work is dissipated, and that idealization follows directly from the two-component strength model of Taylor-Bishop that was discussed earlier. However, test data is usually better fitted by Nova's adaption of the idealization (see Figure 7):

$$\eta_{max} = M_{tc} - (1 - N)\, D_{min} \quad \text{[4 repeated]}$$

This is a 'strength' equation and not a stress-dilatancy rule; M_{tc} and N are experimentally determined properties that, as far as established by present testing, do not vary with void ratio of confining stress. The step made with NS is based on the ideas of Dafalias and co-workers and which is that, while it has been known for at least 50 years that M evolves (see Figure 3), that evolution could reasonably be taken to depend on the state parameter. So, NS uses this idea with the stress-dilatancy rule:

$$D^p = M_i - \eta \quad [44]$$

…where M_i is now something that evolves; in many ways this goes right back to the original views of Taylor-Bishop about the work dissipation function not being a fixed quantity. The evolution is derived by noting that we still want to honour [4] so that [44] must give the same as [4] at peak strength; so:

$$\eta_{max} = M_{tc} - (1 - N)\, D_{min} = M_i - D^p \quad [45]$$

At peak strength dp=0 and dq =0 for all loading paths; so D_{min} = D^p as elastic strain increments vanish at peak strength. Rearranging [45] and adding in equation [9] leads to:

$$M_i = M_{tc} + N\, D_{min} = M_{tc} - N\, \chi\, |\psi| \quad [46]$$

Equation [45] is straightforward for dense soils with ψ<0 as it simply fits widely measured trends in soil behaviour. It is more troublesome for loose soils as the data is unclear on the soil behaviour in this situation. Logically, there are two choices: i) view M_i as something that only arises for dense soils and thus revert to M_i=M_{tc} for loose soil; or, ii) make M_i symmetric around M_{tc} and which then adds the 'modulus' function on ψ. At present the second option appears to fit test data on loose soils better. But, certainly M_i < M_{tc} as the issue is the proportion of plastic work not being dissipated by plastic distortional strain.

This topic of mobilized friction ratio is something requiring an open mind on your part and something to investigate in the soils that you test. There is no 'right' or 'wrong' here, as yet, and that is one reason that there have been variants within NS. But do document the idealization you use when assessing your data.

NS Yield Surface

The yield surface is derived from the normality condition and corresponds to exactly that as done for OCC. But, the NS yield surface, illustrated on Figure 23, differs from that of OCC in two important ways even thought is has the same basic shape.

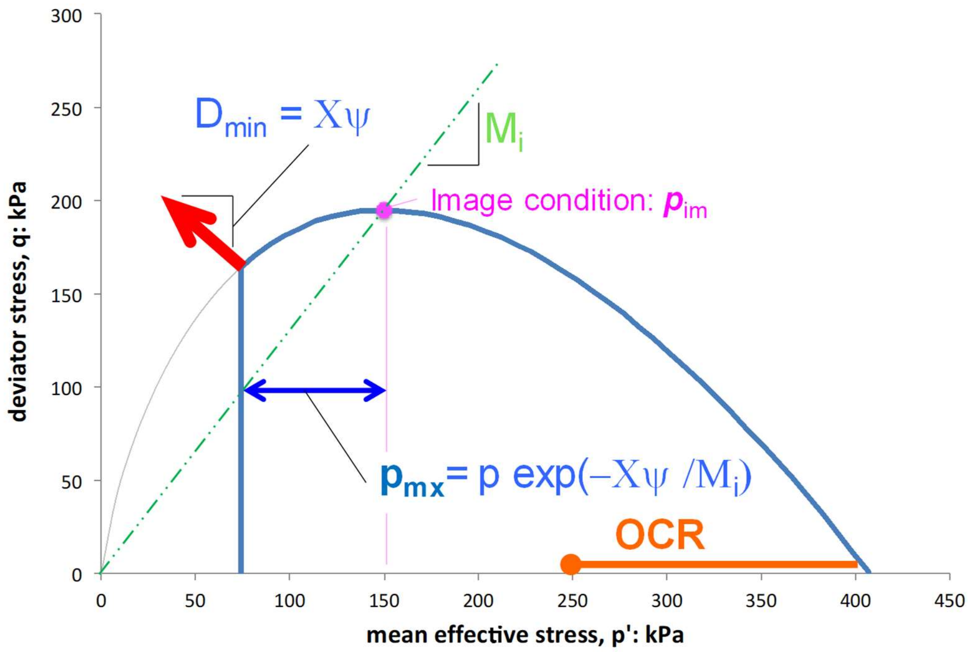


Figure 23: Example of NS yield surface

First, while the yield surface has the 'bullet like' shape of OCC, because the yield surface does not intersect the CSL the scaling parameter is now something different: the image stress p_{im}. At this image condition $D^p = 0$ but this need not be the end point, and usually is not, since the second necessary condition of the critical state that $dD^p/d\varepsilon_q = 0$ is generally not met; it is a bit like a ghostly appearance of the critical state but not quite – hence the name 'image'. Thus, the basic equation for the NS yield surface looks the same as that of OCC except that we have switched out p_c for p_{im}:

$$\eta \,/\, M_i = 1 - \ln(p/p_{im}) \qquad [46]$$

This switch from p_c to p_{im} may appear to be just a change of notation but it is actually fundamental. In OCC there are a very limited range of plastic behaviours with dense soil being viewed as over-consolidated (and easily massively so). In NS, any combination of *e, p, q* can be on a yield surface – this means soil can yield anywhere in its accessible states, exactly as seen when you test soil. Of course, you can unload from a yield surface too so over-consolidation also exists – see Figure 23.

Second, the NS yield surface differs from that of OCC by having an inner cap (the vertical line shown on Figure 23). Recall that state-dilatancy sets a direct relation between the soil's ψ and its limiting dilatancy D_{min} (equation [9]). Since we have normality, this means we can only go so far around a yield surface before we violate the dilatancy limit – and which can also be viewed as limiting how far a yield surface can expand at any mean stress. Thus, we must introduce a hardening limit p_{mx} for p_{im} to control how much dilatancy develops.

The hardening limit is derived by rearranging the NS yield surface equation to get:

$$M_i \ \ln(p/p_{mx}) = M_i - \eta$$

… and then substituting the limiting dilation:

$$M_i \ \ln\left(\frac{p}{p_{mx}}\right) = D_{min} = \ \mathrm{X}\,\psi$$

… and then inverting to get:

$$p_{mx} = p \exp\left(-\mathrm{X}\,\psi/M_i\right) \qquad [47]$$

NS Hardening

The hardening rule in NS comes directly from Axiom 2 of CSSM: everything moves to the CSL with distortional strain. Axiom 2 requires that the 'image condition' (p_{im}) on the NS yield surface moves onto the CSL with deviatoric strain. In NS this is done as 'double hardening' by first moving p_{im} to its limiting condition p_{mx} and then tracking p_{mx} as that evolves to p_c. The second aspect of hardening develops naturally because p_{mx} is based on state-dilatancy (equation [9]). So, it is sufficient to invoke:

$$dp_{im} = H\,(p_{mx} - p_{im})\, d\varepsilon_q^p \qquad [48]$$

…as the NS hardening law where H is the hardening modulus. Because the essence of NS is decoupling of the yield surface from the CSL, the slope of the CSL no longer directly acts as the plastic modulus; generally, an effect of ψ on the hardening modulus is also found. Further, many soils show an effect of softening as the deviator stress ratio increases. Accordingly, the hardening modulus used in NS is:

$$H = \left(H_0 - \ H_\psi\, \psi\right) p \,/\, p_{im} \qquad [49]$$

…where H_0, H_ψ are experimentally determined soil properties. These are the only properties in NS that are specific to NS; determination of their values will be discussed shortly after first describing how to compute NS behaviour for triaxial tests. But, aspects of plastic hardening derived for OCC continue across to NS with an approximation of equation [36] often being a reasonable first-estimate for H_0:

$$H_0 \approx 2\,/\,\lambda \quad \approx 4\,/\,\lambda_{10} \qquad [50]$$

Computing Triaxial compression of NS

Computing drained and undrained triaxial compression behaviour of NS is near identical to the calculations used for OCC other than to replace dp_c/p_c of OCC with dp_{im}/p_{im} for NS. For learning purposes that substitution will be sufficient; for accuracy, an additional term is needed in the consistency condition because M_i evolves:

$$d\eta = M_i\left(\frac{dp_c}{p_c} - \frac{dp}{p}\right) + \eta\,\frac{dM_i}{M_i} \qquad [51]$$

Iterative Modelling for Soil Properties

The essence of all critical state soil mechanics is that soil properties do not change with void ratio or stress level (this may need a little modification, as there is some evidence for an effect of void ratio on M_{tc} with very loose soils). Consistent with this, the properties used in OCC and NS are simply obtained by plotting data from triaxial compression tests with one exception: plastic hardening in NS. There is also the idea that the elastic shear modulus is so sensitive to the details of particle to particle contacts that G_{max} might be reasonably considered as a 'state variable' (which can change from test to test and vary from place to place insitu) rather than as a 'property'. These aspects are easily investigated by *iterative forward modelling* (IFM).

In IFM we estimate the properties of interest and them compute the behaviour for the test in question. We then compare the computed behaviour with that measured, then revising our estimates based on the mismatch between computed and measured. The process is then repeated ("iterated") as many times as needed to achieve good fits to data.

The beauty of IFM is that it optimizes the fit of theory to reality, and while perhaps needing a little more work that determining a property by plotting some data a particular way, it is actually very easy to do using a laptop computer. Indeed, many people find IFM slightly 'sucking them into' the data and continue the modelling looking for ever more refinement; it is actually as involving as a video game.

IFM is used to determine H_0, H_y for NS. However, although this could be done in a worksheet environment, it needs professional programming approach as we want to include things like over-consolidation (very tedious in a worksheet) and the ability to easily jump between the various tests on a soil (almost impossible with worksheets). Thus, the open-code Excel worksheet provided and which you will be guided through as part of Tutorial 5.

Physical Limits on State and Hardening

Although it may seem possible for soil to be as loose as you like with the state parameter approach, there is a physical limit – you cannot have $\eta_{max} < 0$; this criterion can be put in the stress-dilatancy rule to give an upper limit for soil state consistent with its properties:

$$\psi < M_{tc} / (X\,(1+N)) \qquad [52]$$

Likewise, the plastic hardening modulus must be sufficient to the soil cannot 'consolidate' away from the CSL under drained isotropic compression while also ensuring consolidation to at least parallel the CSL in the loosest possible state . Two limitations on the hardening properties follow from this 'physical reasonableness' requirement:

$$H_0 > 1 / \lambda \qquad [53a]$$

$$H_\psi < X\,(H_0 - 1 / \lambda)\,(1+N) / M_{tc} \qquad [53b]$$

These limits are indicated adjacent to the relevant inputs in the modelling environment.

Summary of NS equations

For triaxial compression (see Jefferies & Been, 2016, for generalization to 3D)

Yield surface … or as *Consistency*	$\eta = M_i \left[1 - ln\left(\frac{p_{im}}{p}\right)\right]$ $\Delta\eta = M_i\left(\frac{\Delta p_{im}}{p_{im}} - \frac{\Delta p}{p}\right) + \eta\frac{dM_i}{M_i}$
Flowrule	$D^p = M_i - \eta$ ...where: $M_i = M_c$ *...for simple version* $M_i = M_c - \lvert N \, \mathrm{X} \, \psi \rvert$ *...for accurate version*
Hardening	$\Delta p_{im} = H\,(p_{mx} - p_{im})\,\Delta\varepsilon_q^p$ ...where $p_{mx} = p \exp(-\mathrm{X}\,\psi/M_i)$
Elasticity	$K = 2G(1+\nu)/\,(3-6\nu)$ ...where G, ν are properties

TUTORIAL 4 – Change OCC into NS and validate

Take the worksheet of Exercise 3b with the dense sand test Erksak CID-667 and convert the worksheet from OCC to NS by adding two columns to:

Compute the state parameter ψ

Compute the hardening limit P_{mx}

And then modifying the worksheet to:

Use variable M_i as opposed to constant M

Change the hardening law from that of OCC to NSS

Use H as a simple constant from the inputs

What value for H gives the best-fit to this test on dense sand ?

Why is the computed stress-path not perfect ?

TUTORIAL 5 – Use NS in the VBA modelling environment

Open NorTxl_Nerlerk.xlsm (this is the xls holding the Nerlerk sand data used in the first tutorial and for which you have determined its properties).

Move to 'param & plots' worksheet and learn how to view the different tests by using the 'Plot Data' button.

Check that the NorSand properties are set to those you calibrated.

Now systematically model the ***drained*** tests, varying H as needed, to best-fit all tests; record your simulation parameters as you do so. Plot your results to estimate the plastic hardening modulus for Nerlerk sand. What is this hardening function ? How sensitive is this function to your eyeballed fit ? Should one test be tried a little stiffer, and if so does that improve the fit in your view ?

With this hardening function determined, now model an ***undrained*** test using the measured G_{max} (choose any test and feel free to model a few more). Next, explore the effect of changing G_{max} on the computed stress path. Now optimize the modelled fit to all undrained tests the H-trend from drained tests and using G_{max} as a 'free' parameter that is adjusted to best-fit the computed stress-path to each undrained test in turn. Compare your optimized G_{max} with the expected value based on the geophysical data; can you suggest a 'rule of thumb' for Nerlerk sand elasticity ?

Appendix A: Derivation of state-dilatancy

This appendix considers the mathematics behind the state parameter to derive why maximum dilation, and thus strength, should scale linearly with the state parameter. And, thus to introduce the notion that the state parameter is more than "just another correlation" of soil mechanics.

The early tests by Casagrande identified that sand dilated or contracted as it was sheared, depending on the initial density, to reach a common and unique void ratio at large strain – the critical state for that sand. In modern constitutive developments we take this behaviour as two *axioms* (= fundamental truths that we accept) and construct the theory around them in a mathematically consistent manner. *Axiom 1* is that the CSL is unique; *Axiom 2* is that soil moves to the CSL with distortional strain.

Axiom 2 can be given a simple, and elegant, form by defining a *state parameter* ψ as the void ratio offset from the soil's critical state at the current mean effective stress (e_c):

$$\psi = e - e_c \qquad \text{[A.1]}$$

…and which then leads to *Axiom 2* being stated as:

$$\psi \rightarrow 0 \;\; as \;\; \varepsilon_q \rightarrow \infty \qquad \text{[A.2]}$$

… where ε_q is the deviatoric strain that generalizes to arbitrary 3D loadings as suggested by Resende & Martin (1985). Nothing spurious is introduced by formalizing Casagrande's canonical figure into an equation, and further the equation is a mathematically sufficient.

When faced with a physical process that involves moving from state 'A' to state 'B', a reasonable first estimate is to treat the situation as a *rate process* so that the further you are from the end state the faster you move. Radioactive decay is an example of this. Accordingly, we take [A.2] in a slightly more restrictive form by introducing a first-order rate equation analogous to those found in other branches of physics:

$$d\psi = -\chi' \, \psi \, d\varepsilon_q \qquad \text{[A.3]}$$

… where χ' is a coefficient of proportionality. Notice that the negative sign forces a decrease in deviation, whatever the initial conditions, as deviatoric strain accumulates – thus exactly conforming to [A.2]. Strictly, we could start with 'higher order' terms and simplify to a first approximation. We invoke [A.3] as a fundamental postulate for deriving critical state soil mechanics.

Taking the differential of [A.1] (simple, as it is a linear equation) and substituting in [A.3]:

$$de - de_c = -\chi' \, \psi \, d\varepsilon_q \qquad \text{[A.4]}$$

On dividing [A.4] through by the current specific volume and rearranging:

$$\frac{de}{1+e} = \frac{de_c}{1+e} - \frac{\chi' \, \psi \, d\varepsilon_q}{1+e} \qquad \text{[A.5]}$$

The left-hand term of [A.5] is the volumetric strain increment, so [A.5] re-writes (allowing for the compression positive convention of soil mechanics) as:

$$-d\varepsilon_v = \frac{de}{1+e} = \frac{de_c}{1+e} - \frac{\chi' \psi \; d\varepsilon_q}{1+e} \qquad \text{[A.6]}$$

…or on dividing through by the deviatoric strain:

$$D = \frac{d\varepsilon_v}{d\varepsilon_q} = \frac{\chi' \psi}{1+e} - \frac{1}{1+e}\frac{de_c}{d\varepsilon_q} \qquad \text{[A.7]}$$

…where D is the *dilatancy* of the soil. We can proceed using any admissible form of CSL, but a simple result is obtained if we adopt a semi-log CSL (which is at least a good approximation in most cases, and more complicated CSL can be reduced to this locally):

$$e_c = \Gamma - \lambda \ln p \quad => \quad de_c = -\lambda \frac{dp}{p} \qquad \text{[A.8]}$$

Substituting [A.8] in [A.7] gives:

$$D = \frac{\chi' \psi}{1+e} + \frac{\lambda}{1+e}\frac{1}{p}\frac{dp}{d\varepsilon_q} \qquad \text{[A.9]}$$

Now in a *drained* triaxial test at peak strength $dp/d\varepsilon_q=0$ as the stress path reverses direction for dilating soil. This peak strength corresponds to 'maximum' dilation rate under current stress-dilatancy theories and which is actually D_{min} because of the compression positive convention. Further, because the stress state is stationary at q_{max} there are no elastic strain increments. Thus, our base expectation, derived from *Axiom 2* by introducing a slightly more restrictive postulate of first-order rate theory, is that:

$$D_{min} = D^P_{min} = \frac{\chi' \psi}{1+e} \qquad \text{[A.10]}$$

… in drained triaxial compression. Such a test then becomes a convenient and simple method to determine the coefficient of proportionality in the rate equation – just plot max dilation versus the *state parameter at that dilation*, normalized by specific volume, at that maximum dilation and fit a linear trend to the data that goes through the origin.

Historically, the '*1+e*' term was omitted from the definition of the rate coefficient in equation [A.10], which is unfortunate in principle. However, scatter in soil behaviour data is barely reduced (if at all) if the data is plotted against ψ/(1+e) so, for practical, use it seems sufficient to regard '*1+e*' as "bundled in" to the definition of the rate coefficient – thus the simplification:

$$D_{min} = D^P_{min} = \mathrm{X}\,\psi \qquad \text{[A.11]}$$